PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

A Nonlinear Theory of Water Waves for Finite and Infinite Depths

A. E. Green and P. M. Naghdi

Phil. Trans. R. Soc. Lond. A 1986 **320**, 37-70
doi: 10.1098/rsta.1986.0104

Phil. Trans. R. Soc. Lond. A **320**, 37–70 (1986) [37]
Printed in Great Britain

A NONLINEAR THEORY OF WATER WAVES FOR FINITE AND INFINITE DEPTHS

By A. E. GREEN,[1] F.R.S., and P. M. NAGHDI[2]

[1] *Mathematical Institute, University of Oxford*, 24–29 *St Giles, Oxford OX*1 3*LB, U.K.*

[2] *Department of Mechanical Engineering, University of California, Berkeley, California* 94720, *U.S.A.*

(*Received* 17 *May* 1985)

CONTENTS

This paper is concerned with the construction by a direct approach of a fairly general nonlinear theory of an incompressible inviscid fluid for application to water waves, which upon specialization yields (*a*) a theory suitable for deep waters, (*b*) one for waters of finite (non-shallow) depth and also reduces to (*c*) the theory of a directed fluid sheet given previously by Green & Naghdi (*Proc. R. Soc. Lond.* A **347**, 447–473 (1976*a*); *J. appl. Mech.* **44**, 523–528 (1977)). Whereas our development again is based on a *model* known as the directed- (or Cosserat-) surfaces model, our approach to the subject differs from the earlier one in two respects: (1) the basic conservation equations are recast here in an Eulerian form by means of a procedure utilized recently for viscous fluid flow in channels Green & Naghdi (*Arch. ration. Mech. Analysis* **86**, 39–63 (1984)) and (2) a new procedure for identification of various quantities in the conservation equations which can be specified by constitutive equations. After development of the basic equations in the context of the purely mechanical theory (part A) and their reductions to the three above-mentioned categories for an incompressible, homogeneous, inviscid fluid, the rest of the paper (arranged as parts B, C, and D) deals with applications to various water wave problems under gravity. These include steady-state solutions for surface disturbance of a stream by pressure when the fluid depth may be any one of the above-mentioned three categories and a detailed study of nonlinear stern waves over waters of infinite depth.

PART A. DEVELOPMENTS OF THE BASIC EQUATIONS

1. Introduction

The exact three-dimensional theory of an inviscid fluid, which depends on three space variables and time (and mostly for an incompressible fluid), has long been accepted as a basis for examining the properties of a wide variety of problems of wave propagation in fluids. The nonlinear theory poses considerable analytical difficulties and most attempts at solving problems are based on approximations of the three-dimensional equations, often reducing them to dependence on two space variables and time.

In recent years an alternative direct approach has been proposed, whereby the original intent (for characterization of an approximate theory from the three-dimensional equations) is replaced by an alternative theory; this alternative theory is still three-dimensional in character, but one that depends only on two space variables and time. Clearly, such an alternative theory cannot reflect all the properties covered by the exact three-dimensional theory (neither can approximations to the exact three-dimensional theory) but it is designed to deal with some of its main features. The alternative theory should initially be viewed as one in its own right, although it is of interest to raise the question of what relation it has to the original exact three-dimensional theory. So far, direct theories based on two space variables and time have been proposed for wave problems in waters of relatively small depth, i.e. for *shallow waters*, as well as problems for other types of bodies for which at least one dimension is small.

The present paper is concerned with the formulation of a fairly general nonlinear theory by a direct approach for applications to water waves which, upon specialization, includes those for (*a*) very deep waters, (*b*) waters of finite (non-shallow) depth and (*c*) shallow waters corresponding to the theory of a directed fluid sheet, discussed previously by Green and Naghdi (1976*a*, 1977). In the present work we again use a direct approach based on a *model*, known as the directed (or Cosserat) surfaces model, which, although three-dimensional in character, depends only on two space variables and time. The model comprises a surface and any number of directors, and is motivated partly by the desire to construct a nonlinear theory of water waves

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

for an incompressible, inviscid fluid occupying a semi-infinite region of space instead of the more usual shallow water type theory. The present approach to the subject differs from the earlier one (Green *et al.* 1974; Green & Naghdi 1976*a*, 1977) in two respects: (1) the basic conservation equations are recast here in an Eulerian form by means of a procedure which was used recently for viscous fluid flow problems in channels (Green & Naghdi 1984) and (2) a new procedure for identification of various quantities in the conservation equations, notably the inertia coefficients and velocity fields through suitable weighted averages of their counterparts in the three-dimensional theory.

In the theory of directed fluid sheets, the simplest model is that of a surface with a single director which is constrained to remain parallel to a fixed direction (Green & Naghdi 1977). This particular model satisfies the boundary conditions exactly on the major surfaces of the fluid sheet when surface forces are prescribed, as does the theory in which the single director is not constrained. Additional directors are needed in order to deal with fluid flow problems in which the velocity field is prescribed. For example, for flow of viscous fluid inside channels of arbitrary shape three directors are required in order to satisfy the boundary conditions at the channel walls which may not be stationary (Green & Naghdi 1984).

For an incompressible inviscid fluid under the action of gravity and in the presence of a surface tension, the various coefficients (such as the inertia coefficients) in the case of a directed fluid sheet with a single director are identified by a simple procedure (see, for example, Green & Naghdi 1976*b*) and the resulting system of differential equations for a fluid sheet is expected to be applicable to shallow waters. Clearly, a simple theory of this kind is inappropriate for deep waters or even for waters of (non-shallow) finite depth. For problems in the latter categories the basic theory may be modified by including additional kinematical variables, i.e. additional directors, and hence also additional corresponding kinetical quantities, i.e. additional director forces or couples. Inevitably, such an approach, which is based on the general theory of directed fluid sheets (Green & Naghdi 1976*a*, 1984), can be effected at the expense of considerable complications and would result in a complex system of equations, although these would depend only on two space variables rather than three. To avoid such a complex procedure, we adopt an alternative method which retains the simplicity of a single director or at most two directors with constraints, but identifies the various constitutive coefficients and velocity fields through fairly general weighted averages of their counterparts in the three-dimensional theory.

In order to establish the derived relations for the weighted averages mentioned in the preceding paragraph, we obtain conservation laws from the three-dimensional equations in the Appendix. Section 2 contains a direct formulation of a basic theory in Eulerian form and in a more general manner than that given previously (Green & Naghdi 1976*a*, 1977). This part of our development, which embraces the conservation laws in the purely mechanical theory, is valid for all types of continua. In §3 we discuss the reduction in special forms for incompressible, inviscid fluids of small depth, unlimited depth and (non-shallow) finite depth. The rest of the paper is presented in three parts: B, C and D. In part B (§§4–7) we deal with the theory and applications of water waves under gravity for fluid of infinite depth. Part C (§§8–11) is concerned with the corresponding theory and applications for shallow waters. The theory in part C is the same as that used in earlier papers (e.g. Green & Naghdi 1976*a*, 1977), but most of the applications are new. In part D (§§12–13) a theory and a few applications for waters of (non-shallow) finite depth is presented and the theories for parts B and C in limiting cases are included.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

2. Directed fluid sheets

The theory of directed fluid sheets, or Cosserat surfaces, $\mathscr{C}_K$ – comprising a material surface, $\mathscr{s}$, with K directors $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_K$ – was derived by Green & Naghdi (1976) in Lagrangian form and by Green & Naghdi (1984) in Eulerian form, which is more appropriate for applications to problems in which more than one director is used. Also mention should be made of a theory of Cosserat surfaces (Green & Naghdi 1983), constructed in Lagrangian form, which is capable of more general interpretation. This form of the theory is needed in the present paper and is summarized here in Eulerian form.

Let a fixed surface, $\bar{\mathscr{s}}$, in space be specified by a position vector $\boldsymbol{r}$ which is a function of two curvilinear coordinates $\zeta^\alpha (\alpha = 1, 2)$ on this surface. Base vectors, metric tensors and unit normals on this surface are denoted by $\boldsymbol{a}_\alpha, \boldsymbol{a}^\alpha, a_{\alpha\beta}, a^{\alpha\beta}, \boldsymbol{a}_3$ and are defined in (A 3) of the Appendix. The velocity of material points on the surface, $\mathscr{s}$, of the Cosserat surface, $\mathscr{C}_K$, in its present configuration at time t, is denoted by $\boldsymbol{v} = \boldsymbol{v}(\zeta^\alpha, t)$. Further, in the present configuration at time t the director velocities are designated by $\boldsymbol{w}_M = \boldsymbol{w}_M(\zeta^\alpha, t)$ with $\boldsymbol{w}_0 = \boldsymbol{v}$ and directors $\boldsymbol{d}_M$ assume the values

$$\boldsymbol{d}_M = \bar{\boldsymbol{d}}_M(\zeta^1, \zeta^2), \quad (M = 1, 2, \ldots, K), \quad \boldsymbol{d}_0 = \bar{\boldsymbol{d}}_0 = \boldsymbol{r}(\zeta^1, \zeta^2), \tag{2.1}$$

where the notations $\boldsymbol{w}_0$ and $\bar{\boldsymbol{d}}_0$ are introduced for later convenience.

Throughout the paper, we use standard vector and tensor notations with lower case Latin indices (subscripts or superscripts) taking the values 1, 2, 3 and Greek indices the values 1, 2, together with the usual convention for summation over repeated indices (one subscript and one superscript). Also, we make a comment here in regard to the notation for various quantities associated with $\bar{\mathscr{s}}$. Strictly speaking, the symbol $\bar{\boldsymbol{r}}$ should be used to denote the position vector of $\bar{\mathscr{s}}$, reserving the corresponding symbol without an overbar for the position vector of $\mathscr{s}$; however, for simplicity, since we are concerned with an Eulerian form of the theory, we consistently omit the overbar here and elsewhere in the paper from most of the symbols associated with $\bar{\mathscr{s}}$. A parallel remark applies to the notation for the base vectors $\boldsymbol{a}_\alpha$, metric tensor $a_{\alpha\beta}$, etc., as well as the arbitrary part, $\mathscr{P}$, of $\bar{\mathscr{s}}$ introduced in the next paragraph.

Let $\mathscr{P}$ be an arbitrary part of $\bar{\mathscr{s}}$ bounded by a closed curve $\partial\mathscr{P}$ whose outward unit normal in the surface is

$$\boldsymbol{\nu} = \nu_\alpha \boldsymbol{a}^\alpha = \nu^\alpha \boldsymbol{a}_\alpha. \tag{2.2}$$

Guided by analysis in the Appendix and using a fixed surface area on $\bar{\mathscr{s}}$, we postulate Eulerian forms in the conservation laws for mass, momentum, director momentum and moment of momentum, for the Cosserat surface, $\mathscr{C}_K$, as follows:

$$\frac{\partial}{\partial t}\int_{\mathscr{P}} \rho y_{MN}\, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \rho \boldsymbol{v}_{MN}\cdot\boldsymbol{\nu}\, \mathrm{d}s - \int_{\mathscr{P}} \rho(\boldsymbol{v}_{MN} + \boldsymbol{v}_{NM})\cdot \mathrm{d}\boldsymbol{\sigma} = 0, \tag{2.3}$$

for $M = 0, 1, 2, \ldots, K$; $N = 0, 1, 2, \ldots, K$,

$$\frac{\partial}{\partial t}\int_{\mathscr{P}} \rho \sum_{M=0}^{K} y_{M0}\, \boldsymbol{w}_M\, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \rho \sum_{M=0}^{K} \boldsymbol{w}_M \boldsymbol{v}_{M0}\cdot\boldsymbol{\nu}\, \mathrm{d}s = \int_{\mathscr{P}} \rho \boldsymbol{f}\, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \boldsymbol{n}\, \mathrm{d}s, \tag{2.4}$$

$$\frac{\partial}{\partial t}\int_{\mathscr{P}} \rho \sum_{M=0}^{K} y_{MN}\, \boldsymbol{w}_M\, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \rho \sum_{M=0}^{K} \boldsymbol{w}_M \boldsymbol{v}_{MN}\cdot\boldsymbol{\nu}\, \mathrm{d}s - \int_{\mathscr{P}} \rho \sum_{M=0}^{K} \boldsymbol{w}_M \boldsymbol{v}_{NM}\cdot \mathrm{d}\boldsymbol{\sigma}$$
$$= \int_{\mathscr{P}} (\rho \boldsymbol{l}_N - \boldsymbol{k}_N)\, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \boldsymbol{m}_N\, \mathrm{d}s, \tag{2.5}$$

for $N = 1, 2, \ldots, K$ and

$$\frac{\partial}{\partial t}\int_{\mathscr{P}} \rho \sum_{N=0}^{K} \sum_{M=0}^{K} y_{NM}\, \bar{\boldsymbol{d}}_N \times \boldsymbol{w}_M \, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \rho \sum_{N=0}^{K} \sum_{M=0}^{K} \bar{\boldsymbol{d}}_N \times \boldsymbol{w}_M \boldsymbol{v}_{MN} \cdot \boldsymbol{\nu} \, \mathrm{d}s$$
$$= \int_{\mathscr{P}} \rho \sum_{N=0}^{K} \bar{\boldsymbol{d}}_N \times \boldsymbol{l}_N \, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \sum_{N=0}^{K} \bar{\boldsymbol{d}}_N \times \boldsymbol{m}_N \, \mathrm{d}s. \tag{2.6}$$

The velocity fields $\boldsymbol{v}_{NM}$ and the inertia coefficients y_{NM} which appear in (2.3)–(2.6) are defined in terms of the velocity fields $\boldsymbol{v}$, $\boldsymbol{w}_M$ and the geometry of the three-dimensional body, modelled by the present theory, by (A 11) in the Appendix. Also, $\boldsymbol{n} = \boldsymbol{m}_0$ is the force vector, $\boldsymbol{m}_N$ are the director force vectors at the curve $\partial\mathscr{P}$, $\boldsymbol{f} = \boldsymbol{l}_0$ is the assigned force vector, $\boldsymbol{l}_N$ the assigned director force vectors and $\boldsymbol{k}_N$ are the internal director forces. As explained in previous papers, the assigned force $\boldsymbol{f}$ may be interpreted as the combined effect of (i) the stress vector on the major surfaces of the body, denoted by $\boldsymbol{f}_c$ and (ii) an integrated contribution arising from the three-dimensional body force denoted by $\boldsymbol{f}_b$. A parallel statement holds for the assigned fields $\boldsymbol{l}_N$, so we write

$$\boldsymbol{f} = \boldsymbol{f}_b + \boldsymbol{f}_c, \quad \boldsymbol{l}_N = \boldsymbol{l}_{Nb} + \boldsymbol{l}_{Nc}. \tag{2.7}$$

Since we are limiting attention to problems in which thermal and electromagnetic effects are neglected, the theory is based on the mechanical equations (2.3)–(2.6), valid for all types of continua, fluid or solid.

By usual procedures, from (2.4) and (2.5) we obtain

$$\boldsymbol{n} = \boldsymbol{N}^{\alpha}\nu_{\alpha}, \quad \boldsymbol{m}_M = \boldsymbol{M}^{\alpha}_M \nu_{\alpha}. \tag{2.8}$$

With the use of the two-dimensional analogue of the divergence theorem, as well as (2.8) whenever relevant, the field equations, which result from (2.3)–(2.6), are

$$(\partial/\partial t)\,(\rho y_{MN}) + a^{-\frac{1}{2}}(\rho a^{\frac{1}{2}} \boldsymbol{v}_{MN}\cdot \boldsymbol{a}^{\alpha})_{,\alpha} - \rho(\boldsymbol{v}_{NM} + \boldsymbol{v}_{MN})\cdot \boldsymbol{a}_3 = 0 \tag{2.9}$$

for $M, N = 0, 1, \ldots, K$,

$$\sum_{M=0}^{K} \rho\left\{y_{MO}\frac{\partial \boldsymbol{w}_M}{\partial t} + (\boldsymbol{v}_{MO}\cdot\boldsymbol{a}^{\alpha})\frac{\partial \boldsymbol{w}_M}{\partial \zeta^{\alpha}} + \boldsymbol{w}_M \boldsymbol{v}_{MO}\cdot\boldsymbol{a}_3\right\} = \rho\boldsymbol{f} + \boldsymbol{N}^{\alpha}|_{\alpha}, \tag{2.10}$$

$$\sum_{M=0}^{K} \rho\left\{y_{MN}\frac{\partial \boldsymbol{w}_M}{\partial t} + (\boldsymbol{v}_{MN}\cdot\boldsymbol{a}^{\alpha})\frac{\partial \boldsymbol{w}_M}{\partial \zeta^{\alpha}} + \boldsymbol{w}_M \boldsymbol{v}_{MN}\cdot\boldsymbol{a}_3\right\} = \rho\boldsymbol{l}_N - \boldsymbol{k}_N + \boldsymbol{M}_N{}^{\alpha}|_{\alpha} \tag{2.11}$$

for $N = 1, 2, \ldots, K$ and

$$\bar{\boldsymbol{d}}_{0,\alpha} \times \boldsymbol{N}^{\alpha} + \sum_{N=1}^{K} (\bar{\boldsymbol{d}}_N \times \boldsymbol{k}_N + \bar{\boldsymbol{d}}_{N,\alpha} \times \boldsymbol{M}_N{}^{\alpha}) = \boldsymbol{0}. \tag{2.12}$$

In obtaining (2.12) we have also used (A 20). In these equations $(\ )_{,\alpha} = \partial(\ )/\partial\zeta^{\alpha}$ and a vertical line denotes covariant differentiation with respect to $\bar{\mathfrak{s}}$.

We record here an expression for the rate of work of the external forces acting on a part of the body minus the rate of change of kinetic energy, i.e.

$$\int_{\mathscr{P}} \rho \sum_{N=0}^{K} \boldsymbol{l}_N\cdot\boldsymbol{w}_N \, \mathrm{d}\sigma + \int_{\partial\mathscr{P}} \sum_{N=0}^{K} \boldsymbol{m}_N\cdot\boldsymbol{w}_N \, \mathrm{d}s - \frac{\partial}{\partial t}\int_{\mathscr{P}} \tfrac{1}{2}\rho \sum_{N=0}^{K}\sum_{M=0}^{K} y_{MN}\, \boldsymbol{w}_N\cdot\boldsymbol{w}_M \, \mathrm{d}\sigma$$
$$- \int_{\partial\mathscr{P}} \tfrac{1}{2}\rho \sum_{N=0}^{K}\sum_{M=0}^{K} \boldsymbol{w}_N\cdot\boldsymbol{w}_M \boldsymbol{v}_{MN}\cdot\boldsymbol{\nu}\, \mathrm{d}s = \int_{\mathscr{P}} \left\{\rho \sum_{N=1}^{K} (\boldsymbol{k}_N\cdot\boldsymbol{w}_N + \boldsymbol{M}_N{}^{\alpha}\cdot\boldsymbol{w}_{N,\alpha}) + \rho\boldsymbol{N}^{\alpha}\cdot\boldsymbol{v}_{,\alpha}\right\}\mathrm{d}\sigma. \tag{2.13}$$

To obtain the right-hand side of (2.13) we have used (2.9)–(2.11).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

3. Reduction to special forms

The direct mechanical theory postulated in §2, and based on three-dimensional integral balance laws valid for all bodies either solid or fluid, contains vector and tensor functions which have three-dimensional components but which depend only on two space variables and the time. To complete the theory specific constitutive equations are required for the (three-dimensional) response functions, $\boldsymbol{N}^{\alpha}$, $\boldsymbol{k}_N$, $\boldsymbol{M}_N{}^{\alpha}$, which represent the material properties of the body and its particular geometry. In addition, the inertia coefficients, y_{MN}, and the relation of the velocity fields, $\boldsymbol{v}_{MN}$, to the director velocities, $\boldsymbol{w}_M$, must be specified, which can usually be done with the help of (A 11). Also the theory must then be interpreted in relation to the motion of a three-dimensional body and its geometry. This involves an identification of the weighting functions, $\lambda_N(\zeta)$, which are used in the Appendix to motivate the form of the balance laws chosen in §2. The aim is to have the simplest form of the general theory which will represent in a significant way the main characteristics of the motion. In previous forms of the theory, because of the type of boundary conditions often used on the major surfaces of the body, it was sufficient to use a moving surface with one director corresponding to a weighting function ζ. For viscous fluid flow in a channel with fixed or moving walls, it was necessary to have a moving surface with at least three directors corresponding to weighting functions ζ, ζ^2, ζ^3. In all these applications the choice of the weighting functions implied that the theory was limited to shell like bodies in which one dimension of the body was small in some sense. In the present form of the theory with a more general interpretation of the theory via weighting functions $\lambda_N(\zeta)$, it is possible to remove the limitation of applications only to shell-like bodies. In particular, we wish to be able to use the theory for the propagation of nonlinear waves over inviscid fluids of infinite and large depths, which is our main purpose here.

We restrict further attention to homogeneous incompressible inviscid fluids and divide the rest of the paper into three parts: B, C and D.

Part B is concerned with wave propagation on fluids of infinite depth. For this we use a moving surface and two directors which corresponds to weighting functions $e^{a\zeta}$ and ζ, where a is a parameter which will be specified by a constitutive equation. Also the velocity field $\boldsymbol{v} = \boldsymbol{w}_0$ is constrained at time t and the velocity $\boldsymbol{w}_2$ is constrained to be zero.

Part C deals with wave propagation on fluids of small depth. The theory is already available from previous papers but is rederived here directly in Eulerian form using a moving surface and a single director constrained to remain parallel to a vertical direction and corresponding to a weighting function ζ. New applications are made to problems similar to those considered in part B, for completeness, and comparisons with part B are made.

In *Part D* we consider wave propagation on fluids of uniform finite depth which is not necessarily either small or very large. We use a moving surface and directors which now correspond to weighting functions $\cosh a(\zeta+h)$, $\sinh a(\zeta+h)$ and ζ, where h is a constant and a is a parameter to be specified by a constitutive equation. The velocity field $\boldsymbol{v}$ is constrained to be constant at time t, the second director is constrained to move vertically, and the velocity of the third director is constrained to be zero.

PART B. FLUIDS OF INFINITE DEPTH

4. Introduction

We consider an inviscid, incompressible fluid of constant density ρ^* and infinite depth. We use the theory of §2 in which $\bar{\sigma}$ is a plane surface $\zeta = x_3 = 0$, where x_i $(i = 1, 2, 3)$ is a rectangular Cartesian coordinate system with the x_3 direction being vertical. Then, we may set

$$\zeta^\alpha = x_\alpha, \quad \boldsymbol{a}_\alpha = \boldsymbol{a}^\alpha = \boldsymbol{e}_\alpha, \quad \boldsymbol{a}_3 = \boldsymbol{e}_3, \quad a_{\alpha\beta} = a^{\alpha\beta} = \delta_{\alpha\beta}, \tag{4.1}$$

where $\boldsymbol{e}_i$ is a constant orthonormal set of vectors associated with x_i. The fluid is subject to constant gravity g in the direction $-\boldsymbol{e}_3$ (not to be confused with g of the Appendix which here is 1) and bounded by the surface

$$\zeta = x_3 = \beta(x_1, x_2, t), \tag{4.2}$$

at which there is a pressure $\hat{p} = \hat{p}(x_1, x_2, t)$ and a constant surface tension T. The fluid occupies the region $-h < \zeta \leqslant \beta$ with h allowed to approach infinity, i.e. $h \to \infty$. From §2 we select the theory with two directors which correspond to weighting functions $\lambda_1(\zeta) = \mathrm{e}^{a\zeta}$ and $\lambda_2(\zeta) = \zeta$. In addition

$$\bar{\boldsymbol{d}}_0 = x_\alpha \boldsymbol{e}_\alpha + \mathrm{d}\boldsymbol{e}_3, \quad \bar{\boldsymbol{d}}_1 = \boldsymbol{0}, \quad \bar{\boldsymbol{d}}_2 = \boldsymbol{e}_3, \tag{4.3}$$

where the scalar d is a constant. The velocity $\boldsymbol{v}$ is constrained always to be constant in a direction parallel to the plane $\bar{\sigma}$ and the velocity $\boldsymbol{w}_2$ is constrained to be zero, so that

$$\boldsymbol{v} = c_\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{w}_2 = \boldsymbol{0}, \tag{4.4}$$

where c_α are constants. Since the various quantities in §2 both here and in the remainder of the paper are referred to base vectors $\boldsymbol{e}_i$, the distinction between subscripts and superscripts disappears and summation is implied over indices at the same level, usually subscripts. Referred to the basis $\boldsymbol{e}_i$, the velocity field $\boldsymbol{w}_1$ can be written as

$$\boldsymbol{w}_1 = w_i \boldsymbol{e}_i, \tag{4.5}$$

where the index 1, denoting the first director, is omitted in the components for convenience.

In view of (A 19), the incompressibility condition is

$$w_{\alpha,\alpha} + a w_3 = 0. \tag{4.6}$$

Because of the constraint of incompressibility (4.6) and the constrained values of $\boldsymbol{v}$ and $\boldsymbol{w}_2$ in (4.4), the response functions $\boldsymbol{N}_\alpha$, $\boldsymbol{k}_1$, $\boldsymbol{M}_{1\alpha}$, $\boldsymbol{k}_2$, $\boldsymbol{M}_{2\alpha}$ contain constraint responses

$$\boldsymbol{N}_\alpha = \boldsymbol{r}_\alpha = r_{i\alpha} \boldsymbol{e}_i, \quad \boldsymbol{k}_1 = -pa\boldsymbol{e}_3, \quad \boldsymbol{M}_{1\alpha} = -p\boldsymbol{e}_\alpha, \quad \boldsymbol{M}_{2\alpha} = \bar{r}_{i\alpha} \boldsymbol{e}_i, \quad \boldsymbol{k}_2 = r_i \boldsymbol{e}_i, \tag{4.7}$$

where p, $\bar{r}_{i\alpha}$, r_i and $r_{i\alpha}$ are arbitrary functions of x_1, x_2, t. The constraint responses are found with the help of the mechanical power in (2.13) and are such that the corresponding mechanical power is zero. Since the fluid is incompressible there is no additional contribution to the response functions which have the final form (4.7). These satisfy (2.12) if

$$r_{3\alpha} = r_\alpha, \quad r_{\alpha\beta} = r_{\beta\alpha}. \tag{4.8}$$

In view of (A 11) and (A 17) we choose the following values for y_{MN}, $\boldsymbol{v}_{MN}$, $\boldsymbol{f}$, $\boldsymbol{l}_1$:

$$
\begin{aligned}
\rho y_{00} &= \rho^*(\beta+h), \quad \rho y_{10} = (\rho^*/a)\,\mathrm{e}^{a\beta}, \quad \rho y_{11} = (\rho^*/2a)\,\mathrm{e}^{2a\beta},\\
\rho\boldsymbol{v}_{00} &= \{\rho^* c_\alpha(\beta+h) + (\rho^*/a)\,w_\alpha\,\mathrm{e}^{a\beta}\}\boldsymbol{e}_\alpha,\\
\rho\boldsymbol{v}_{01} &= (\rho^*/a)\,(c_\alpha\,\mathrm{e}^{a\beta} + \tfrac{1}{2}w_\alpha\,\mathrm{e}^{2a\beta})\,\boldsymbol{e}_\alpha,\\
\rho\boldsymbol{v}_{10} &= (\rho^*/a)\,(c_\alpha\,\mathrm{e}^{a\beta} + \tfrac{1}{2}w_\alpha\,\mathrm{e}^{2a\beta})\,\boldsymbol{e}_\alpha + \tfrac{1}{2}\rho^* w_3\,\mathrm{e}^{2a\beta}\,\boldsymbol{e}_3,\\
\rho\boldsymbol{v}_{11} &= (\rho^*/a)\,(\tfrac{1}{2}c_\alpha\,\mathrm{e}^{2a\beta} + \tfrac{1}{3}w_\alpha\,\mathrm{e}^{3a\beta})\,\boldsymbol{e}_\alpha + \tfrac{1}{3}\rho^* w_3\,\mathrm{e}^{3a\beta}\,\boldsymbol{e}_3,\\
\rho\boldsymbol{f} &= (\hat{p}-q)\,\beta_{,\alpha}\,\boldsymbol{e}_\alpha - \{\hat{p}-q-\bar{p}+\rho^* g(\beta+h)\}\,\boldsymbol{e}_3,\\
\rho\boldsymbol{l}_1 &= (\hat{p}-q)\,\mathrm{e}^{a\beta}\,\beta_{,\alpha}\,\boldsymbol{e}_\alpha - \{(\hat{p}-q)\,\mathrm{e}^{a\beta} + \rho^*(g/a)\,\mathrm{e}^{a\beta}\}\,\boldsymbol{e}_3,
\end{aligned}
\tag{4.9}
$$

where $h\to\infty$ after substitution of these results in the field equations, $\bar{p}$ is the pressure at the bottom of the stream which may tend to infinity and q represents the effect of surface tension T given by

$$q = \frac{T\{[1+(\beta_{,1})^2]\,\beta_{,22} - 2\beta_{,1}\beta_{,2}\beta_{,12} + [1+(\beta_{,2})^2]\,\beta_{,11}\}}{[1+(\beta_{,1})^2+(\beta_{,2})^2]^{\frac{3}{2}}}. \tag{4.10}$$

With the help of (4.7)–(4.9), (2.9)–(2.11) reduce to

$$\partial\beta/\partial t + (c_\alpha + w_\alpha\,\mathrm{e}^{a\beta})\,\partial\beta/\partial x_\alpha - w_3\,\mathrm{e}^{a\beta} = 0, \tag{4.11}$$

$$(\rho^*/2a)\,\mathrm{e}^{2a\beta}\,\partial w_\alpha/\partial t + (\rho^*/a)\,(\tfrac{1}{2}\mathrm{e}^{2a\beta}c_\lambda + \tfrac{1}{3}\mathrm{e}^{3a\beta}w_\lambda)\,\partial w_\alpha/\partial x_\lambda + \tfrac{1}{3}\rho^*\,\mathrm{e}^{3a\beta}\,w_3 w_\alpha = -\partial p/\partial x_\alpha + (\hat{p}-q)\,\mathrm{e}^{a\beta}\,\partial\beta/\partial x_\alpha, \tag{4.12}$$

$$(\rho^*/2a)\,\mathrm{e}^{2a\beta}\,\partial w_3/\partial t + (\rho^*/a)\,(\tfrac{1}{2}\mathrm{e}^{2a\beta}c_\lambda + \tfrac{1}{3}\mathrm{e}^{3a\beta}w_\lambda)\,\partial w_3/\partial x_\lambda + \tfrac{1}{3}\rho^*\mathrm{e}^{3a\beta}\,w_3^2 = ap - (\rho^* g/a)\,\mathrm{e}^{a\beta} - (\hat{p}-q)\,\mathrm{e}^{a\beta}. \tag{4.13}$$

The equations involving the response functions $r_{i\alpha}$, $\bar{r}_{i\alpha}$, r_i are omitted as they are not required in the rest of part B.

Before considering further the nonlinear system of (4.6) and (4.11)–(4.13) it is instructive to examine the linearized form of these equations when wave heights β are small and equilibrium surface pressure is zero. These are

$$
\begin{aligned}
& w_{\alpha,\alpha} + a w_3 = 0,\\
& \partial\beta/\partial t + c_\alpha\,\partial\beta/\partial x_\alpha - w_3 = 0,\\
& (\rho^*/2a)\,(\partial w_\alpha/\partial t + c_\lambda\,\partial w_\alpha/\partial x_\lambda) = -\partial p/\partial x_\alpha,\\
& (\rho^*/2a)\,(\partial w_3/\partial t + c_\lambda\,\partial w_3/\partial x_\lambda) = ap - (\rho^* g/a)\,(1+a\beta) - \hat{p} + T\,\partial^2\beta/\partial x_\alpha\,\partial x_\alpha.
\end{aligned}
\tag{4.14}
$$

One example of the use of these equations is for wave propagation in the x_1-direction under constant surface pressure and surface tension, of the form $\beta = L\sin m(x-ct)$, $c_\lambda = 0$, or equivalently standing waves $\beta = L\sin mx$ on a uniform stream $c_1 = c$. Either is a possible solution of (4.14) provided that

$$c^2 = 2ag\{(1+Tm^2/(\rho^* g)\}/(a^2+m^2). \tag{4.15}$$

If we now choose $a = m$ this reduces to

$$c^2 = (g/m)\{1 + Tm^2/(\rho^* g)\}, \tag{4.16}$$

which is the same as the value found from the linearized three-dimensional theory (see, for example, Lamb 1932, §267).

We return to the nonlinear system (4.6) and (4.11)–(4.13), and consider travelling-wave solutions in the x_1-direction or, equivalently, steady-motion waves on a stream moving with constant speed c in the x_1-direction. Then $c_1 = c$, $c_2 = 0$, $w_2 = 0$ and β, w_1, w_3 and p are functions of x_1. We assume that the surface pressure $\hat{p}$ is a constant p_0 and neglect surface tension. It follows from (4.6) and (4.11) that

$$w_1 = c(A - a\beta)\,\mathrm{e}^{-a\beta}, \quad w_3 = c(1 + A - a\beta)\,\mathrm{e}^{-a\beta}\,\beta', \tag{4.17}$$

where A is a constant and a prime denotes differentiation with respect to x_1. The x_2 component of (4.12) is satisfied and, with the help of (4.17), the other equation in (4.12) yields

$$p/\rho^* = (p_0/\rho^* a)\,\mathrm{e}^{a\beta} + B + (c^2/2a)\,(2 + A - a\beta)\,\mathrm{e}^{a\beta}, \tag{4.18}$$

where B is a constant. With the help of (4.17), again, (4.13) gives

$$p\,\mathrm{e}^{-a\beta}/\rho^* = p_0/(\rho^* a) + g/a^2 + c^2(1 + A - a\beta)\,(3 + 2A - 2a\beta)\,\beta''/(6a^2) - c^2(4 + 3A - 3a\beta)\,\beta'^2/(6a). \tag{4.19}$$

By eliminating p between (4.18) and (4.19) and performing one integration, we have

$$\tfrac{1}{12}\Gamma\,(\beta')^2(1 + A - a\beta)^2\,(3 + 2A - 2a\beta) = \tfrac{1}{2}\Gamma\{(1 + A)\,(2 + A)\,a\beta - \tfrac{1}{2}(3 + 2A)\,a^2\beta^2 + \tfrac{1}{3}a^3\beta^3\}$$
$$- (1 + A)\,a\beta + \tfrac{1}{2}a^2\beta^2 - D(A - a\beta)\,\mathrm{e}^{-a\beta} + E, \tag{4.20}$$

where E is a constant and

$$\Gamma = ac^2/g, \quad D = Ba^2/g. \tag{4.21}$$

The properties of nonlinear progressive waves of finite amplitude on deep water, or the equivalent properties of standing waves on a uniform stream moving with speed c, may be discussed with the aid of (4.17)–(4.20). Before proceeding further with this we consider the possibility of finite amplitude waves of the Stokes type. It is well known (see, for example, Whitham (1974), §13.13) that finite amplitude waves of this type can be found as approximate solutions of the three-dimensional equations, although Benjamin & Feir (1967) have shown that such waves are unstable. We note here that in the present theory, based on (4.17)–(4.20), with surface tension neglected, approximate solutions of the Stokes type may be found in the form

$$\beta = b\cos mx + \tfrac{1}{2}mb^2\cos 2mx + \ldots,$$
$$c^2 = (g/m)\,(1 + b^2m^2 + \ldots) \tag{4.22}$$

with a again chosen to be m as in the linearized theory. As far as the approximation is taken, the results (4.22) agree with those found from the three-dimensional equations.

The type of wave motion predicted by (4.20) depends on the choice of constants of

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

integration, A, D and E, and these depend on the boundary conditions. As an example, a simple case of a standing wave on a uniform stream of speed c, occurs when

$$A = 0, \quad D = 0, \quad E = 0, \quad \Gamma = \tfrac{9}{8}. \tag{4.23}$$

Then (4.20) becomes

$$3(\beta')^2 = a\beta(1-2a\beta)(4-3a\beta)/\{(1-a\beta)^2(3-2a\beta)\}, \tag{4.24}$$

which represents a standing wave with minimum and maximum heights corresponding to $\beta = 0$, $\beta = 1/(2a) = 4c^2/(9g) = h^*$, respectively. If λ is the wavelength, then

$$\frac{\lambda}{h^*} = 4(3)^{\frac{1}{2}}\int_0^{\frac{1}{2}} \frac{(1-\xi)(3-2\xi)^{\frac{1}{2}}\,d\xi}{\{\xi(1-2\xi)(4-3\xi)\}^{\frac{1}{2}}}. \tag{4.25}$$

As a second example, suppose A, D and E are chosen so that when $\beta = 0$

$$w_i = 0, \quad \beta' = 0, \quad \beta'' = 0 \ (\beta = 0), \tag{4.26}$$

which means that as $x_1 \to \pm\infty$ there is just a uniform stream with speed c. Then, from (4.17) and (4.20) it follows that

$$A = 0, \quad E = 0, \quad D = 1-\Gamma, \tag{4.27}$$

and (4.20) becomes

$$\tfrac{1}{12}\Gamma(1-a\beta)^2(3-2a\beta)(\beta')^2 = a\beta\{(\Gamma-1)(1-e^{-a\beta}) - \tfrac{1}{4}(3\Gamma-2)\,a\beta + \tfrac{1}{6}\Gamma a^2\beta^2\}. \tag{4.28}$$

The right-hand side of (4.28) has a double zero at $a\beta = 0$, a simple zero at $a\beta = \gamma < 1$, and is positive in between, provided that

$$6(e-2)/(5e-12) > \Gamma > 2. \tag{4.29}$$

Then (4.28) represents a stationary solitary wave on a stream of constant speed c (or equivalently a solitary wave moving with speed c on still water). The maximum wave height is $\bar{h} = \gamma/a$ and, recalling (4.21), the speed is

$$c = (g\Gamma\bar{h}/\gamma)^{\frac{1}{2}}, \tag{4.30}$$

where Γ lies in the range (4.29); the number γ depends, of course, on Γ. The corresponding speed of a solitary wave on a fluid of initial depth h, as computed by Boussinesq (1871), Rayleigh (1876) and by Green *et al.* (1974) from a theory of the form given later in §8, is $\{g(\bar{h}+h)\}^{\frac{1}{2}}$.

5. Surface disturbance of a stream by pressures: infinite depth

We consider here the steady two-dimensional flow of a stream moving in the x-direction on which is imposed the following surface pressures:

$$p = \begin{cases} p_0 & -\infty < x \leqslant x_1, \quad x_2 < x < \infty, \\ p_1 & x_1 \leqslant x \leqslant x_2, \end{cases} \tag{5.1}$$

where p_0, p_1 are constants and x_1, x_2 are positions on the x-axis, not to be confused with coordinates x_1, x_2 used earlier in the paper. We divide the fluid into three regions as follows: region I, $x \leqslant x_1$; region II, $x_1 \leqslant x \leqslant x_2$ and region III, $x_2 \leqslant x$.

In region I,

$$w_1 = c(A_1 - a\beta)\, e^{-a\beta}, \quad w_3 = c(1 + A_1 - a\beta)\, e^{-a\beta}\, \beta', \tag{5.2a}$$

$$\tfrac{1}{12}\Gamma(\beta')^2\,(1+A_1-a\beta)^2\,(3+2A_1-2a\beta) = \tfrac{1}{2}\Gamma\{(1+A_1)\,(2+A_1)\,a\beta - \tfrac{1}{2}(3+2A_1)\,a^2\beta^2 + \tfrac{1}{3}a^3\beta^3\}$$
$$-(1+A_1)\,a\beta + \tfrac{1}{2}a^2\beta^2 - D_1(A_1 - a\beta)\,e^{-a\beta} + E_1, \tag{5.2b}$$

$$p/\rho^* = (p_0/\rho^* a)\,e^{a\beta} + (c^2/2a)\,(2+A_1-a\beta)\,e^{a\beta} + gD_1/a^2. \tag{5.2c}$$

In region II,

$$w_1 = c(A_2 - a\beta)\, e^{-a\beta}, \quad w_3 = c(1 + A_2 - a\beta)\, e^{-a\beta}\, \beta', \tag{5.3a}$$

$$\tfrac{1}{12}\Gamma(\beta')^2(1+A_2-a\beta)^2(3+2A_2-2a\beta) = \tfrac{1}{2}\Gamma\{(1+A_2)\,(2+A_2)\,a\beta - \tfrac{1}{2}(3+2A_2)\,a^2\beta^2 + \tfrac{1}{3}a^3\beta^3\}$$
$$-(1+A_2)\,a\beta + \tfrac{1}{2}a^2\beta^2 - D_2(A_2 - a\beta)\,e^{-a\beta} + E_2, \tag{5.3b}$$

$$p/\rho^* = (p_1/\rho^* a)\,e^{a\beta} + (c^2/2a)\,(2+A_2-a\beta)\,e^{a\beta} + g\,D_2/a^2. \tag{5.3c}$$

In region III, equations similar in form to (5.2*a*)–(5.2*c*) but with A_1, D_1, E_1 replaced by A_3, D_3, E_3.

At $x = x_1$, x_2 we suppose that the wave heights are $\beta = \beta_1, \beta_2$ respectively and at $x \to -\infty$, $\beta = 0$ and the stream moves with constant speed c, with $\beta' = \beta'' = 0$. Then, from (5.2*a*)–(5.2*c*) it follows that

$$A_1 = 0, \quad E_1 = 0, \quad D_1 = 1 - \Gamma. \tag{5.4}$$

From (5.2*b*) it is then seen that the type of motion in region I depends on the values of Γ which may be $\Gamma \gtrless 2$.

We now seek a solution of the flow problem in which β, β', p and mass flow are continuous at $x = x_1$, x_2. This yields the values

$$A_2 = 0, \quad A_3 = 0, \quad D_2 = 1 - \Gamma - \mu\, e^{a\beta_1}, \quad E_2 = \mu a\beta_1,$$
$$D_3 = 1 - \Gamma + \mu\,(e^{a\beta_2} - e^{a\beta_1}), \quad E_3 = \mu(a\beta_1 - a\beta_2), \tag{5.5}$$

where $p_1 - p_0 = \rho^* g\mu/a$. Then, in region II

$$\tfrac{1}{12}\Gamma(\beta')^2(1-a\beta)^2(3-2a\beta) = \Gamma(a\beta - \tfrac{3}{4}a^2\beta^2 + \tfrac{1}{6}a^3\beta^3) - a\beta + \tfrac{1}{2}a^2\beta^2$$
$$+ (1 - \Gamma - \mu\, e^{a\beta_1})\, a\beta\, e^{-a\beta} + \mu a\beta_1, \tag{5.6}$$

and in region III,

$$\tfrac{1}{12}\Gamma(\beta')^2\,(1-a\beta)^2\,(3-2a\beta) = \Gamma(a\beta - \tfrac{3}{4}a^2\beta^2 + \tfrac{1}{6}a^3\beta^3) - a\beta + \tfrac{1}{2}a^2\beta^2$$
$$+ \{1 - \Gamma + \mu\,(e^{a\beta_2} - e^{a\beta_1})\}\, a\beta\, e^{-a\beta} + \mu(a\beta_1 - a\beta_2). \tag{5.7}$$

If $\beta_1 > \beta_2$ and $\Gamma \geqslant 2$, then no satisfactory wave motion is possible in region III. We next consider the case $\Gamma < 2$. From (5.2*b*), with the help of (5.4), it is then seen that throughout region I, $\beta = 0$ everywhere and hence $\beta_1 = 0$. In order to gain some insight into the nature of the solution we first consider the case when the pressure p_1 applied over the region $x_1 \leqslant x \leqslant x_2$ differs slightly from the remaining surface pressure p_0, i.e. when μ is small. Then, in region II we have

$$a\beta = -\{2\mu/(2-\Gamma)\}\{1 - \cos m(x - x_1)\},$$
$$a\beta_2 = -\{2\mu/(2-\Gamma)\}\{1 - \cos m(x_2 - x_1)\}. \tag{5.8}$$

and in region III,

$$a\beta = -\{2\mu/(2-\Gamma)\}\cos m(x - x_2) + \{2\mu/(2-\Gamma)\}\cos m(x - x_1), \tag{5.9}$$

where
$$m^2/a^2 = (2-\Gamma)/\Gamma. \tag{5.10}$$

A value for the coefficient Γ must still be specified. The results obtained for small values of μ may be compared with those derived by Lamb (1932, §24) using the linear three-dimensional theory of inviscid fluid flow for fluids of infinite depths. Lamb shows that the surface elevation downstream of the applied pressure consists of two parts. One part diminishes rapidly as we move downstream of the pressure, but is infinite at the edge of the pressure, whereas the second part consists of a harmonic wave which is identical to (5.9) if we choose

$$\Gamma = 1, \quad a = m = g/c^2. \tag{5.11}$$

This means that the parameter a (or equivalently Γ) has a constitutive equation in terms of the wavelength of the downstream wave which should be a guide for determining Γ in the nonlinear problem.

We leave aside the discussion of the nonlinear case and consider the special problem of an isolated force acting at the point $x = x_1$. In this case, $x_2 - x_1 \to 0$ and $\mu \to \infty$, $\beta_2 \to 0$, in such a way that $\mu(x_2 - x_1)$ is finite. Then it follows from (5.6) that

$$x_2 - x_1 \to \Gamma^{\frac{1}{2}}\bar{H}/(\mu\bar{H}a)^{\frac{1}{2}}, \quad \beta_2 = -\bar{H} \to 0. \tag{5.12}$$

The pressure acts along the normal to the surface over the interval $x_2 - x_1$ so that, in the limit, the isolated force has vertical and horizontal components N and L, respectively, at $x = x_1$ given by

$$\left.\begin{aligned} N &\to \rho^* g\mu(x_2 - x_1)/a \to \rho^* c^2 (\mu\bar{H}/a\Gamma)^{\frac{1}{2}}, \\ L &\to \rho^* g\mu\bar{H}/a \to \rho^* c^2(\mu\bar{H})/\Gamma. \end{aligned}\right\} \tag{5.13}$$

If, in addition, $\mu(x_2 - x_1)$ and hence N and L are small, then L may be neglected compared with N and, from (5.7) or (5.9), the surface elevation in region III is given approximately by

$$\beta = -2\{\Gamma/(2-\Gamma)\}^{\frac{1}{2}}(N/\rho^* c^2)\sin mx, \tag{5.14}$$

where m is defined by (5.10). Again, this may be compared with results in Lamb if we make the choice (5.11) for a (or Γ), apart from the part of the solution in Lamb which is infinite at the origin and diminishes rapidly as we move from the isolated force.

For the isolated force problem, (5.7) in region III becomes

$$\tfrac{1}{12}\Gamma(\beta')^2(1-a\beta)^2(3-2a\beta) = \Gamma(a\beta - \tfrac{3}{4}a^2\beta^2 + \tfrac{1}{6}a^3\beta^3) - a\beta + \tfrac{1}{2}a^2\beta^2 + (1-\Gamma)\,a\beta\,\mathrm{e}^{-a\beta} + (\mu\bar{H}a)\,(1 - a\beta\,\mathrm{e}^{-a\beta}). \tag{5.15}$$

In general this equation will yield periodic wave solutions depending on the value of $\mu\bar{H}a$ and of Γ. We omit a detailed discussion but note that, for small values of $\mu\bar{H}a$, a periodic wave may be found in series form whose first two terms are

$$\beta = -2(N/\rho^* c^2)\sin mx + (N/\rho^* c^2)^2\, m\{4\sin^2 mx - \tfrac{14}{3}(1-\cos mx)\} + \ldots \quad (\text{where } m = g/c^2) \tag{5.16}$$

and
$$\Gamma = 1 + \ldots.$$

To complete the discussion of the surface disturbance of a stream by a band of constant pressure on a part of the surface we ask what happens when $\Gamma > 2$. In this case the character of the flows in the three regions differs from that considered previously. Now we assume that in region III as $x \to \infty$ the stream has a constant speed c and that $\beta = 0$, $\beta' = 0$, $\beta'' = 0$. Application of these conditions and the continuity conditions at $x = x_1, x_2$ to the sets of equations (5.2)–(5.3), with a similar set for region III, yields the values

$$\begin{aligned} &A_3 = 0, \quad E_3 = 0, \quad D_3 = 1-\Gamma \\ &A_2 = 0, \quad E_2 = \mu a\beta_2, \quad D_2 = 1-\Gamma-\mu\,\mathrm{e}^{a\beta_2}, \\ &A_1 = 0, \quad E_1 = \mu a(\beta_2-\beta_1), \quad D_1 = 1-\Gamma-\mu\,\mathrm{e}^{a\beta_2}+\mu\mathrm{e}^{a\beta_1}, \end{aligned} \tag{5.17}$$

and we finally obtain the results: in region III,

$$\tfrac{1}{12}\Gamma(\beta')^2(1-a\beta)^2(3-2a\beta) = \Gamma(a\beta-\tfrac{3}{4}a^2\beta^2+\tfrac{1}{6}a^3\beta^3)-a\beta+\tfrac{1}{2}a^2\beta^2+(1-\Gamma)\,a\beta\,\mathrm{e}^{-a\beta}; \tag{5.18}$$

in region II,

$$\tfrac{1}{12}\Gamma(\beta')^2(1-a\beta)^2(3-2a\beta) = \Gamma(a\beta-\tfrac{3}{4}a^2\beta^2+\tfrac{1}{6}a^3\beta^3)-a\beta+\tfrac{1}{2}a^2\beta^2 + (1-\Gamma)\,a\beta\,\mathrm{e}^{-a\beta}+\mu a\beta_2-\mu a\beta\,\mathrm{e}^{a(\beta_2-\beta)}; \tag{5.19}$$

in region I,

$$\tfrac{1}{12}\Gamma(\beta')^2(1-a\beta)^2(3-2a\beta) = \Gamma(a\beta-\tfrac{3}{4}a^2\beta^2+\tfrac{1}{6}a^3\beta^3)-a\beta+\tfrac{1}{2}a^2\beta^2 + (1-\Gamma)\,a\beta\,\mathrm{e}^{-a\beta}-\mu a(\beta_1-\beta_2)+\mu\,(\mathrm{e}^{a\beta_1}-\mathrm{e}^{a\beta_2})\,a\beta\,\mathrm{e}^{-a\beta}. \tag{5.20}$$

We limit attention here to the special case of an isolated force acting at $x = x_1$ so that $\mu \to \infty$, $x_1-x_2 \to 0$, $\beta_1-\beta_2 \to 0$ and $\mu(x_1-x_2)$ is finite. We put $\beta_1 = \beta_2(1+\epsilon)$ where $\epsilon \to 0$ with $\mu\epsilon$ being finite. Then, from (5.19) we have

$$x_2-x_1 \to (2\epsilon\beta_2/\mu\epsilon M)\,\{L^{\frac{1}{2}}-(L-\mu\epsilon M)^{\frac{1}{2}}\}, \tag{5.21}$$

where

$$\begin{aligned} \tfrac{1}{12}L\Gamma(1-a\beta_2)^2(3-2a\beta_2) &= \Gamma(a\beta_2-\tfrac{3}{4}a^2\beta_2^2+\tfrac{1}{6}a^3\beta_2^3)-a\beta_2+\tfrac{1}{2}a^2\beta_2^2+(1-\Gamma)\,a\beta_2\,\mathrm{e}^{-a\beta_2}, \\ \tfrac{1}{12}M\Gamma(1-a\beta_2)(3-2a\beta_2) &= a\beta_2. \end{aligned} \tag{5.22}$$

Also, the components N, $\bar{L}$ of the isolated force are given by

$$\begin{aligned} &N \to \rho^* g\mu(x_2-x_1)/a \to 2\rho^* c^2\beta_2\{L^{\frac{1}{2}}-(L-\mu\epsilon M)^{\frac{1}{2}}\}/(M\Gamma), \\ &\bar{L} \to \rho^* c^2(\mu\epsilon)\,\beta_2/\Gamma, \end{aligned} \tag{5.23}$$

and (5.20) becomes

$$\tfrac{1}{12}(\beta')^2(1-a\beta)^2(3-2a\beta) = \Gamma(a\beta-\tfrac{3}{4}a^2\beta^2+\tfrac{1}{6}a^3\beta^3)-a\beta+\tfrac{1}{2}a^2\beta^2 + (1-\Gamma)\,a\beta\,\mathrm{e}^{-a\beta}-\mu\epsilon a\beta_2(1-a\beta\,\mathrm{e}^{a(\beta_2-\beta)}). \tag{5.24}$$

If Γ lies in the range

$$2 < \Gamma < 6(\mathrm{e}-2)/(5\mathrm{e}-12),$$

with some restrictions on the values of β_2 and $\mu\epsilon$, (5.24) will represent periodic wave motion in region I upstream of the applied force. This is similar to wave motion ahead of a pressure disturbance on a stream of small depth discussed in §9.

Other motions associated with the problem of a pressure disturbance on a uniform stream are also possible. These involve waves in both the upstream and downstream regions of the applied pressure but we do not discuss these here.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

6. Gliding on the surface of a stream: infinite depth

Problems of steady two-dimensional planing on water of large depth have been considered by Wagner (1932), Squire (1957) and Cumberbatch (1958) using the linearized three-dimensional equations of an incompressible inviscid fluid and small planing angles. The special problem of two-dimensional planing of a flat plate on a stream of large, or finite depth, when gravity effects are neglected has been examined by Green (1935, 1936 *a*, *b*, 1938) using classical nonlinear inviscid fluid theory. On the other hand the nonlinear steady state solution to the problem of the transition to planing of a two-dimensional boat, has been solved by Naghdi & Rubin (1981 *a*) on the basis of the nonlinear theory given in §8 for streams of small depth and for Froude numbers $F < 1$, where $F^2 = c^2/(gh)$, h is the upstream depth of the fluid and c its speed.

In the present section we study some aspects of the special problem of the steady state planing of a flat plate on a stream of large depth using the nonlinear theory of §4. Again, we divide the fluid into three regions: I $(x < x_1)$; II $(x_1 \leqslant x \leqslant x_2)$; III $(x_2 < x)$. In regions I and III the surface pressure $\hat{p}$ is constant, equal to p_0, whereas in region II the surface has the prescribed shape $\beta = (x_1 - x)\tan\alpha$ with the leading edge of the plate at height $b\tan\alpha$ above the trailing edge, where $b = l\cos\alpha$ and l is the length of the plate. Equations (4.11)–(4.13) may be partly integrated in the form (4.19) and (4.20) for regions I, III. A partial integration for region II under the plate may also be effected when the surface has the prescribed shape $\beta = (x_1 - x)\tan\alpha$. Thus, we have: in region I,

$$w_1 = c(A_1 - a\beta)\,\mathrm{e}^{-a\beta}, \quad w_3 = c(1 + A_1 - a\beta)\,\mathrm{e}^{-a\beta}\beta', \tag{6.1a}$$

$$\begin{aligned}\tfrac{1}{12}\Gamma(\beta')^2(1 + A_1 - a\beta)^2(3 + 2A_1 - 2a\beta) = \tfrac{1}{2}\Gamma\{&(1 + A_1)(2 + A_1)a\beta - \tfrac{1}{2}(3 + 2A_1)a^2\beta^2 \\ &+ \tfrac{1}{3}a^3\beta^3\} - (1 + A_1)a\beta + \tfrac{1}{2}a^2\beta^2 - D_1(A_1 - a\beta)\,\mathrm{e}^{-a\beta} + E_1,\end{aligned} \tag{6.1b}$$

$$p/\rho^* = (p_0/\rho^* a)\,\mathrm{e}^{a\beta} + (c^2/2a)(2 + A_1 - a\beta)\,\mathrm{e}^{a\beta} + gD_1/a^2; \tag{6.1c}$$

in region II,

$$w_1 = c(A_2 - a\beta)\,\mathrm{e}^{-a\beta}, \quad w_3 = c(1 + A_2 - a\beta)\,\mathrm{e}^{-a\beta}\beta', \tag{6.2a}$$

$$\begin{aligned}p/\rho^* = {}&(c^2/2a)\{2 + A_2 + (1 + A_2)a\beta - \tfrac{1}{2}a^2\beta^2\}\,\mathrm{e}^{a\beta} - (g/a^2)a\beta\,\mathrm{e}^{a\beta} \\ &+ (c^2/6a)\{(4 + 3A_2)a\beta - 3a^2\beta^2/2\}\,\mathrm{e}^{a\beta}\tan^2\alpha + E_2\,\mathrm{e}^{a\beta},\end{aligned} \tag{6.2b}$$

$$\begin{aligned}\hat{p}/\rho^* = {}&(c^2/2)\{2 + A_2 + (1 + A_2)a\beta - \tfrac{1}{2}a^2\beta^2\} - (g/a)(1 + a\beta) \\ &+ (c^2/6)\{4 + 3A_2 + (1 + 3A_2)a\beta - 3a^2\beta^2/2\}\tan^2\alpha + aE_2;\end{aligned} \tag{6.2c}$$

in region III,

$$w_1 = c(A_3 - a\beta)\,\mathrm{e}^{-a\beta}, \quad w_3 = c(1 + A_3 - a\beta)\,\mathrm{e}^{-a\beta}\beta', \tag{6.3a}$$

$$\begin{aligned}\tfrac{1}{12}\Gamma(\beta')^2(1 + A_3 - a\beta)^2(3 + 2A_3 - 2a\beta) = \tfrac{1}{2}\Gamma\{&(1 + A_3)(2 + A_3)a\beta - \tfrac{1}{2}(3 + 2A_3)a^2\beta^2 + \tfrac{1}{3}a^3\beta^3\} \\ &- (1 + A_3)a\beta + \tfrac{1}{2}a^2\beta^2 - D_3(A_3 - a\beta)\,\mathrm{e}^{-a\beta} + E_3,\end{aligned} \tag{6.3b}$$

$$p/\rho^* = (p_0/\rho^* a)\,\mathrm{e}^{a\beta} + (c^2/2a)(2 + A_3 - a\beta)\,\mathrm{e}^{a\beta} + gD_3/a^2. \tag{6.3c}$$

An application of these equations to the plate problem would now follow the same pattern as that used by Naghdi & Rubin (1981 *a*, 1984) for small depths. As shown by them, the solution for $F < 1$ in which flow ahead of the plate tends to that of a uniform stream involves a discontinuity in the slope of the surface at the leading edge of the plate. As a result it is necessary to discuss jump conditions at this edge because an isolated force and couple act there to maintain this discontinuity. A solution of this type is not possible for $F > 1$ and this aspect of the problem was not considered further by Naghdi & Rubin. A similar situation holds for plates on fluids of infinite depth. Here we leave aside those aspects of the problem analogous to those discussed by Naghdi & Rubin and only consider the flows corresponding to high speeds ($F > 1$). Thus, we assume that in the downstream region III as $x \to \infty$ the stream flows with constant speed c and $\beta = \beta' = \beta'' = 0$. The height of the trailing edge of the plate at $x = x_2$ above the level of the stream as $x \to \infty$ is β_2, and the leading edge $x = x_1$ is at a height $\beta_1 = \beta_2 + b\tan\alpha$. In view of the conditions as $x \to \infty$, it follows from (6.3) that

$$A_3 = 0, \quad E_3 = 0, \quad D_3 = 1 - \Gamma, \quad \Gamma = c^2 a/g > 2. \tag{6.4}$$

We now impose continuity of mass flow, β, β', p at $x = x_1, x_2$ so that

$$A_1 = 0, \quad A_2 = 0$$

$$\begin{aligned}
&\tfrac{1}{12}\Gamma(1-a\beta_2)^2(3-2a\beta_2)\tan^2\alpha = \Gamma(a\beta_2 - \tfrac{3}{4}a^2\beta_2^2 + \tfrac{1}{6}a^3\beta_2^3) - a\beta_2 + \tfrac{1}{2}a^2\beta_2^2 + D_3 a\beta_2\,\mathrm{e}^{-a\beta_2},\\
&\quad p_0/(\rho^* a) + gD_3\,\mathrm{e}^{-a\beta_2}/a^2 = (c^2/a)(a\beta_2 - \tfrac{1}{4}a^2\beta_2^2) - (g/a^2)\,a\beta_2\\
&\qquad\qquad + (c^2/6a)(4a\beta_2 - 3a^2\beta_2^2/2)\tan^2\alpha + E_2,\\
&\tfrac{1}{12}\Gamma(1-a\beta_1)^2(3-2a\beta_1)\tan^2\alpha = \Gamma(a\beta_1 - \tfrac{3}{4}a^2\beta_1^2 + \tfrac{1}{6}a^3\beta_1^3) - a\beta_1 + \tfrac{1}{2}a^2\beta_1^2 + D_1 a\beta_1\,\mathrm{e}^{-a\beta_1} + E_1,\\
&\quad p_0/(\rho^* a) + gD_1\,\mathrm{e}^{-a\beta_1}/a^2 = (c^2/a)(a\beta_1 - \tfrac{1}{4}a^2\beta_1^2) - (g/a^2)\,a\beta_1\\
&\qquad\qquad + (c^2/6a)(4a\beta_1 - 3a^2\beta_1^2/2)\tan^2\alpha + E_2.
\end{aligned}\tag{6.5}$$

We continue the discussion only for small values of the planing angle α so that (6.5) may be solved approximately for β_1, β_2, D_1, E_1, E_2. Thus,

$$\begin{aligned}
&a\beta_2 = \{\Gamma/(\Gamma-2)\}^{\frac{1}{2}}\tan\alpha, \quad a\beta_1 = [ab + \{\Gamma/(\Gamma-2)\}^{\frac{1}{2}}]\tan\alpha,\\
&E_2 = p_0/(\rho^* a) - g(\Gamma-1)/a^2 - \tfrac{1}{4}g(\Gamma-2)\beta_2^2 = p_0/(\rho^* a) - g(\Gamma-1)/a^2 - (g\Gamma\tan^2\alpha)/4a^2,\\
&D_1 = 1 - \Gamma + \tfrac{1}{4}(\Gamma-2)(a^2\beta_1^2 - a^2\beta_2^2) = 1 - \Gamma + \tfrac{1}{4}(\Gamma-2)\,ab\,[ab + 2\{\Gamma/(\Gamma-2)\}^{\frac{1}{2}}]\tan^2\alpha,\\
&E_1 = -\tfrac{1}{4}(\Gamma-2)(a^2\beta_1^2 - a^2\beta_2^2) = -\tfrac{1}{4}(\Gamma-2)\,ab\,[ab + 2\{\Gamma/(\Gamma-2)\}^{\frac{1}{2}}]\tan^2\alpha.
\end{aligned}\tag{6.6}$$

With the help of (6.2c) and (6.6) the net total lift, L, normal to the plate is found to be

$$L = \int_{\beta_2}^{\beta_1} \frac{\hat{p} - p_0}{\sin\alpha}\,\mathrm{d}\beta = \frac{\rho^* c^2 l(\Gamma-2)}{4\Gamma}\left\{ab + 2\left(\frac{\Gamma}{\Gamma-2}\right)^{\frac{1}{2}}\right\}\tan\alpha \tag{6.7}$$

approximately, while (6.1b) in region I becomes

$$\begin{aligned}
\tfrac{1}{12}\Gamma(\beta')^2(1-a\beta)^2(3-2a\beta) &= \Gamma(a\beta - \tfrac{3}{4}a^2\beta^2 + \tfrac{1}{6}a^3\beta^3) - a\beta + \tfrac{1}{2}a^2\beta^2\\
&\quad + (1-\Gamma)\,a\beta\,\mathrm{e}^{-a\beta} - \tfrac{1}{4}(\Gamma-2)(a^2\beta_1^2 - a^2\beta_2^2)(1 - a\beta\,\mathrm{e}^{-a\beta}).
\end{aligned}\tag{6.8}$$

Let F be the Froude number defined in terms of half the length of the plate. Then,

$$F^2 = 2c^2/(gl) = 2\Gamma/(al)$$

and from (6.6) and (6.7) we have

$$2\beta_2/l = F^2 \tan\alpha/[\Gamma(\Gamma-2)]^{\frac{1}{2}}, \tag{6.9a}$$

and

$$2L/(\rho^* c^2 l) = [(\Gamma-2)/F^2+\{(\Gamma-2)/\Gamma\}^{\frac{1}{2}}] \tan\alpha \tag{6.9b}$$

$$\tfrac{1}{12}\Gamma(\beta')^2(1-a\beta)^2(3-2a\beta) = \Gamma(a\beta-\tfrac{3}{4}a^2\beta^2+\tfrac{1}{6}a^3\beta^3)-a\beta+\tfrac{1}{2}a^2\beta^2+(1-\Gamma)\,a\beta\,\mathrm{e}^{-a\beta} \\ -(\Gamma/F^2)\,[\Gamma(\Gamma-2)/F^2+\{\Gamma(\Gamma-2)\}^{\frac{1}{2}}]\,(1-a\beta\,\mathrm{e}^{-a\beta})\tan^2\alpha \tag{6.10}$$

approximately. Equation (6.10) represents a periodic wave motion with approximate minimum and maximum heights given by

$$l\{1+F^2/[\Gamma(\Gamma-2)]^{\frac{1}{2}}\}^{\frac{1}{2}}\tan\alpha \tag{6.11}$$

and

$$\frac{l\gamma F^2}{2\Gamma}-\frac{l(1-\gamma\,\mathrm{e}^{-\gamma})\,[\{\Gamma(\Gamma-2)/F^2\}+\{\Gamma(\Gamma-2)\}^{\frac{1}{2}}]\tan^2\alpha}{2(1-\gamma)\,\{(\Gamma-1)\,(\mathrm{e}^{-\gamma}-1)+\frac{1}{2}\gamma\Gamma\}}, \tag{6.12}$$

respectively, where $\gamma < 1$ is the positive root of the equation

$$(\Gamma-1)\,(1-\mathrm{e}^{-\delta})-\tfrac{1}{4}(3\Gamma-2)\,\delta+\tfrac{1}{6}\Gamma\delta^2 = 0, \tag{6.13}$$

and

$$2 < \Gamma < 6(\mathrm{e}-2)\,(5\mathrm{e}-12). \tag{6.14}$$

For given values of F and for a series of values of Γ in the range (6.14) we may compute β_2/l and $L/(\rho^* c^2 l)$ from (6.9) and hence obtain $L/(\rho^* c^2 l)$ as a function of β_2/l for each F.

The foregoing model for gliding of a plate is likely to be unsuitable when gravity effects are small compared with inertia effects, i.e. for large values of F. It is then appropriate to use the theory in Green (1936*b*).

As in the problem of a pressure distribution on a uniform stream, other motions which involve waves in both regions upstream and downstream of the plate are possible but we do not deal with these here.

7. Nonlinear stern waves: infinite depth

Steady two-dimensional flow of an inviscid fluid past a semi-infinite flat-bottomed body has been studied by Vanden-Broeck (1980) using the three-dimensional equations for an inviscid fluid, the fluid having infinite depth. Since very few solutions of the three-dimensional nonlinear equations are available it is of interest to see if the theory of §4 for fluid of infinite depth can also be used to study this problem.

As in §4, we again consider an incompressible inviscid fluid of infinite depth. For steady two-dimensional flow (in the vertical plane), we assign labels I and II to the regions $-\infty < x \leqslant 0$ and $0 \leqslant x < \infty$, respectively. In region I, the fluid is bounded above by a fixed boundary $\beta = 0$, the stream speed is c and the pressure at the fixed boundary is p_1. In region II the upper surface is free and at constant pressure p_0 and leaves the fixed surface at $x = 0$ smoothly. The upper boundary in region I represents the bottom of a flat-bottomed body of draft H so that

$$p_1 - p_0 = \rho^* g H. \tag{7.1}$$

Then, from §4, for steady motion we have: in region I,

$$\beta = 0, \quad p = \rho^* g/a^2 + p_1/a \tag{7.2}$$

and in region II:

$$w_1 = c(A - a\beta)\,\mathrm{e}^{-a\beta}, \quad w_2 = c(1 + A - a\beta)\,\mathrm{e}^{-a\beta}\beta',$$

$$\tfrac{1}{12}\Gamma(\beta')^2(1+A-a\beta)^2(3+2A-2a\beta) = \tfrac{1}{2}\Gamma\{(1+A)(2+A)\,a\beta - \tfrac{1}{2}(3+2A)\,a^2\beta^2 + \tfrac{1}{3}a^3\beta^3\}$$
$$-(1+A)\,a\beta + \tfrac{1}{2}a^2\beta^2 - D(A-a\beta)\,\mathrm{e}^{-a\beta} + E,$$

$$p/\rho^* = (p_0/\rho^* a)\,\mathrm{e}^{a\beta} + (c^2/2a)(2+A-a\beta)\,\mathrm{e}^{a\beta} + gD/a^2. \qquad (7.3\,a\text{–}d)$$

At $x = 0$, mass flow, β, β' and p are continuous so that

$$A = 0, \quad E = 0, \quad D = 1 - \Gamma + aH, \qquad (7.4)$$

where (7.1) has also been used. Setting

$$\eta = a\beta, \quad \bar{x} = ax, \quad \Phi = 1/\Gamma, \quad F^2 = c^2/(gH) = \Gamma/(aH), \qquad (7.5\,a\text{–}d)$$

where F is the Froude number defined in terms of the draft H, (7.3*c*) in region II becomes

$$(\eta')^2(1-\eta)^2(1-2\eta/3) = 4\eta\{1 - \Phi - \tfrac{1}{4}(3-2\Phi)\,\eta + \tfrac{1}{6}\eta^2 - (1-\Phi-1/F^2)\,\mathrm{e}^{-\eta}), \qquad (7.6)$$

where a prime denotes differentiation with respect to $\bar{x}$. The right-hand side of (7.6) has a simple zero $\eta = \eta_1$ if

$$1 - \Phi - \tfrac{1}{4}(3-2\Phi)\,\eta_1 + \tfrac{1}{6}\eta_1^2 - (1-\Phi-1/F^2)\,\mathrm{e}^{-\eta_1} = 0, \quad \Phi = 1/\Gamma. \qquad (7.7)$$

We require that this zero η_1 is < 1 which imposes restrictions on the values of Φ. Then (7.6) represents a periodic wave motion in region II whose peak-to-trough wave height, h, is

$$h/H = \eta_1 \Phi F^2. \qquad (7.8)$$

The wavelength, λ, is given by

$$\frac{\lambda}{2\pi F^2 H} = \frac{\Phi}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}u}{G^{\frac{1}{2}}}, \qquad (7.9)$$

where

$$G(u)\,(1-\eta\sin^2 u)^2(1-\tfrac{2}{3}\eta_1\sin^2 u) = \tfrac{1}{4}(3-2\Phi) - \tfrac{1}{6}\eta_1(1+\sin^2 u)$$
$$+\left\{\frac{(1-\Phi-1/F^2)}{\eta_1\cos^2 u}\right\}(\mathrm{e}^{-\eta_1} - \mathrm{e}^{-\eta_1\sin^2 u}). \qquad (7.10)$$

To complete the problem we must specify a constitutive equation for Φ which is assumed to be a function of F. Here we specify Φ (or equivalently the non-dimensional coefficient aH), by choosing the wavelength, λ, in (7.9) to be given by

$$\lambda = k\pi/a \quad \text{or} \quad \lambda/H = k\pi\Phi F^2, \qquad (7.11)$$

where k is a constant. Hence, from (7.9)

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\mathrm{d}u}{G^{\frac{1}{2}}} = \tfrac{1}{2}k. \qquad (7.12)$$

The steepness, s, of the wave is

$$s = h/\lambda = \eta_1/k\pi. \qquad (7.13)$$

If $\Phi \to k_0$ when $F^2 \to \infty$, it is seen from (7.7) and (7.8) that

$$\eta_1 \to 4/\{(2k_0 - 1)\,F^2\}, \quad h/H \to 4k_0/(2k_0-1). \qquad (7.14)$$

Also, from (7.12), (7.9) and (7.13)

$$k = 2/(2k_0-1)^{\frac{1}{2}}, \quad \lambda/(2\pi F^2 H) \to k_0/(2k_0-1)^{\frac{1}{2}}, \quad s\pi F^2 \to 2/(2k_0-1)^{\frac{1}{2}}. \tag{7.15}$$

In order to illustrate the general character of the solution it is sufficient for our purpose to obtain asymptotic expansions for h and λ in terms of F. It is, however, slightly more convenient to express the solution in terms of η_1, which is the root of (7.7) lying between 0 and 1. Thus, we set

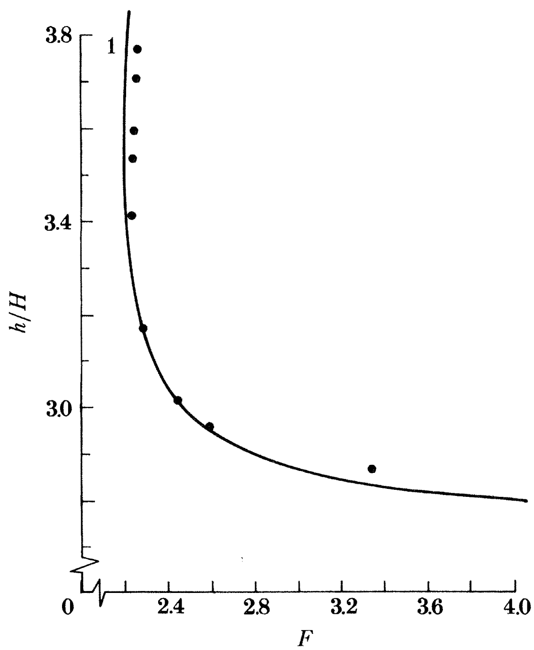
$$\Phi = k_0 - k_1\eta_1 - k_2\eta_1^2 - \ldots, \quad F^{-2} = m_1\eta_1 + m_2\eta_1^2 + m_3\eta_1^3 + \ldots \tag{7.16}$$

and substitute these expressions into (7.7) and (7.12) which must then be satisfied identically. This yields the values

$$\left.\begin{aligned} k_0 &= \tfrac{1}{2} + 2/k^2, \quad k_1 = \tfrac{1}{4} + 5/3k^2, \quad k_2 = -\tfrac{1}{768}k^2 - \tfrac{5}{288} - 83/144k^2, \\ m_1 &= 1/k^2, \quad m_2 = -\tfrac{1}{24} - 5/6k^2, \quad m_3 = \tfrac{1}{1536}k^2 + \tfrac{5}{576} + 35/288k^2 \end{aligned}\right\} \tag{7.17}$$

for the earlier coefficients in the series (7.16). Evaluation of further coefficients is a somewhat lengthy algebraic process so we restrict our computations to (7.16) with the values (7.17) and for values of η_1 between 0 and 0.75. It is likely that accuracy, compared with a complete solution, will be less satisfactory for values of η_1 at the upper end of the range. It remains to specify the value of the coefficient k. For this we may get some guidance from the solution of the problem given by Vanden-Broeck (1980) who uses three-dimensional inviscid fluid theory. We observe that Vanden-Broeck gives the following asymptotic values for h/H, λ/H:

$$h/H \to 2^{\frac{3}{2}}, \quad \lambda/(2\pi HF^2) \to 1 \tag{7.18a, b}$$



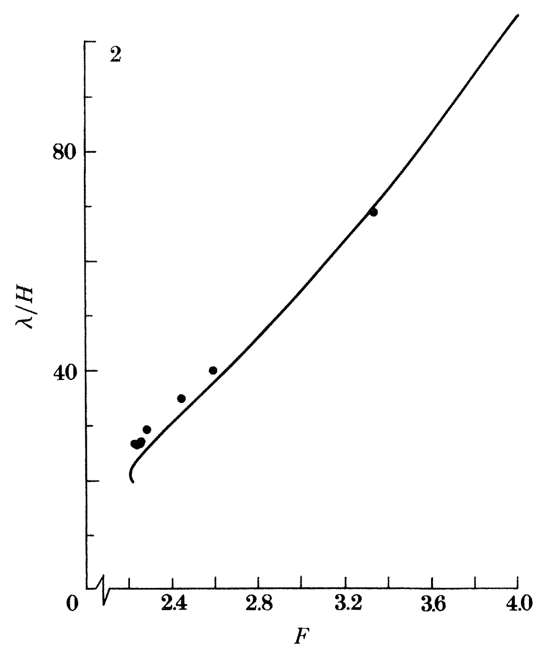


FIGURE 1. A plot of the ratio of the wave height, h, to the draft, H, of the flat-bottomed body against the Froude number, F (defined by (7.5d)). Also shown are the values (–•–) from table 1 of Vanden-Broeck's (1980) exact solution.

FIGURE 2. A plot of the ratio of wavelength λ to the draft, H, of the flat-bottomed body against F. Also shown are the values (–•–) from table 1 of Vanden-Broeck's (1980) exact solution.

as $F \to \infty$. It is clear from (7.14) and (7.15) that we cannot obtain *both* of these values in the present theory by suitable choice of k (or k_0). For example, we may take $k^2 = 2$ (or $k_0 = 1$) and then (7.15*b*) gives the value 1 as in (7.18). Then, however, (7.14) yields $h/H \to 4$ which does not compare very favourably with (7.18*a*). With some guidance also from the remaining values of h and λ for varying values of F, we choose $k^2 = \frac{3}{2}$. Then, for large values of F,

$$\lambda/(2\pi F^2 H) \to 1.123, \quad h/H \to 2.75. \tag{7.19}$$

This gives a difference of less than 3% in the value of h/H when compared with (7.18) and a difference of 12% in $\lambda/(2\pi F^2 H)$.

With $k^2 = \frac{3}{2}$, the values of Φ and $1/F^2$ have been found from (7.16) and (7.17) and then h/H and λ/H have been computed from (7.8) and (7.11).

It is found that the solution exists only for values of $F \gtrsim 2.20$ and that in the neighbourhood of this value of F, both h/F and λ/F are multivalued. Vanden-Broeck found that his solution exists for values of $F \geqslant 2.23$ and also noted multivaluedness. Figures 1 and 2 represent plots of h/H and λ/H as functions of F, where comparisons with the results of Vanden-Broeck (as presented in his table 1) are also indicated.

PART C. FLUIDS OF SMALL DEPTH

8. Introduction

Based on the theory of §2 we recover here the theory of inviscid fluid flow for small depths h, directly in Eulerian form, instead of from the Lagrangian equations given in previous papers. The fluid has constant density ρ^* and in the theory of §2, $\bar{\mathscr{s}}$ is a plane surface $\zeta = x_3 = 0$ and we use the notation of (4.1). The fluid is subject to constant gravity g in the direction $-\boldsymbol{e}_3$ and is bounded by the fixed surface $x_3 = 0$ and a surface

$$\zeta = x_3 = \phi(x_1, x_2, t) \tag{8.1}$$

at which there is a pressure $\hat{p} = \hat{p}(x_1, x_2, t)$ and a constant surface tension T. From §2 we select the theory with one director which corresponds to a choice of weighting function $\lambda_1(\zeta) = \zeta$ in the Appendix. Referred to the basis $\boldsymbol{e}_i$, the velocity vector $\boldsymbol{v}$, director velocity vector $\boldsymbol{w}_1$, and the various kinetical quantities in §2 can be expressed as

$$\begin{gathered}\boldsymbol{v} = v_\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{w}_1 = w_i \boldsymbol{e}_i, \quad \bar{\boldsymbol{d}}_1 = \boldsymbol{e}_3,\\ \boldsymbol{N}_\alpha = N_{i\alpha} \boldsymbol{e}_i, \quad \boldsymbol{M}_{1\alpha} = M_{1i\alpha} \boldsymbol{e}_i,\\ \boldsymbol{k}^1 = k^1_i \boldsymbol{e}_i, \quad \boldsymbol{f} = f_i \boldsymbol{e}_i, \quad \boldsymbol{l}^1 = l^1_i \boldsymbol{e}_i.\end{gathered} \tag{8.2}$$

Since the fluid is incompressible it follows that

$$v_{\alpha,\alpha} + w = 0, \tag{8.3}$$

where $w_3 = w$. Now, let the director be constrained such that

$$w_\alpha = 0. \tag{8.4}$$

The constraint responses are found with the help of the mechanical power (2.13) and are such that the corresponding mechanical power is zero, and are

$$\boldsymbol{N}_\alpha = -p\boldsymbol{e}_\alpha, \quad \boldsymbol{k}^1 = -p\boldsymbol{e}_3 + r_\beta \boldsymbol{e}_\beta, \quad \boldsymbol{M}_{1\alpha} = r_{\alpha\beta} \boldsymbol{e}_\beta, \tag{8.5}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where p, r_β and $r_{\alpha\beta}$ are arbitrary functions of x_α, t. Since the fluid is inviscid (8.5) represents the complete specification for the response functions, and in view of (2.12)

$$r_\beta = 0. \tag{8.6}$$

In view of (A 11) and (A 17) we choose the following values for y_{MN}, $\boldsymbol{v}_{MN}$, $\boldsymbol{f}$, $\boldsymbol{l}^1$:

$$\begin{gathered}\rho y_{00} = \rho^*\phi, \quad \rho y_{10} = \tfrac{1}{2}\rho^*\phi^2, \quad \rho y_{11} = \tfrac{1}{3}\rho^*\phi^3,\\ \rho\boldsymbol{v}_{00} = \rho^*\phi v_\alpha \boldsymbol{e}_\alpha, \quad \rho\boldsymbol{v}_{10} = \tfrac{1}{2}\rho^*\phi^2(v_\alpha\boldsymbol{e}_\alpha + w\boldsymbol{e}_3),\\ \rho\boldsymbol{v}_{01} = \tfrac{1}{2}\rho^*\phi^2 v_\alpha\boldsymbol{e}_\alpha, \quad \rho\boldsymbol{v}_{11} = \tfrac{1}{3}\rho^*\phi^3(v_\alpha\boldsymbol{e}_\alpha + w\boldsymbol{e}_3),\\ \rho\boldsymbol{f} = -\rho^* g\phi\boldsymbol{e}_3 + (\hat{p}-q)\,\phi_{,\alpha}\boldsymbol{e}_\alpha - (\hat{p}-q-\bar{p})\,\boldsymbol{e}_3,\\ \rho\boldsymbol{l}^1 = -\tfrac{1}{2}\rho^* g\phi^2\boldsymbol{e}_3 + (\hat{p}-q)\,\phi\phi_{,\alpha}\boldsymbol{e}_\alpha - (\hat{p}-q)\,\phi\boldsymbol{e}_3,\end{gathered} \tag{8.7}$$

where $\bar{p}$ is the pressure at the fixed surface $x_3 = 0$ and q is given by a formula of the type (4.10) with β replaced by ϕ. With the help of (8.3)–(8.7) the relevant field equations in (2.9)–(2.11) reduce to

$$\begin{gathered}\partial\phi/\partial t + (\phi v_\alpha)_{,\alpha} = 0,\\ \rho^*\phi(\partial v_\alpha/\partial t + v_\beta\,\partial v_\alpha/\partial x_\beta) = -\partial p/\partial x_\alpha + (\hat{p}-q)\,\partial\phi/\partial x_\alpha,\\ \tfrac{1}{2}\rho^*\phi^2(\partial w/\partial t + v_\beta\,\partial w/\partial x_\beta + w^2) = -\rho^* g\phi - (\hat{p}-q-\bar{p}),\\ \tfrac{1}{3}\rho^*\phi^3(\partial w/\partial t + v_\beta\,\partial w/\partial x_\beta + w^2) = p - \tfrac{1}{2}\rho^* g\phi^2 - (\hat{p}-q)\,\phi.\end{gathered} \tag{8.8}$$

The equations involving the constraint response functions $r_{\alpha\beta}$ are omitted since these are not required. The results (8.3) and (8.8) are equivalent to those found in a number of previous papers from a Lagrangian formulation.

Steady state one-dimensional solutions of equations (8.3) and (8.8), in which all functions depend only on x_1, are possible if

$$\begin{gathered}v_1 = k/\phi, \quad w = k\phi'/\phi^2,\\ p/\rho^* = p_0\phi/\rho^* + R\phi - \tfrac{1}{2}g\phi^2 - \tfrac{1}{2}k^2(1+\tfrac{1}{3}(\phi')^2)/\phi,\\ \hat{p}/\rho^* = p_0/\rho^* + R - g\phi - \tfrac{1}{2}k^2(1-\tfrac{1}{3}(\phi')^2)/\phi^2 - \tfrac{1}{3}k^2\phi''/\phi,\end{gathered} \tag{8.9}$$

where k, R, p_0 are constants, a prime denotes differentiation with respect to x_1 and surface tension is omitted. Moreover, if $\hat{p} = p_0$, a constant, then

$$\tfrac{1}{3}k^2(\phi')^2 = k^2 - 2S\phi + 2R\phi^2 - g\phi^3, \quad p/\rho^* = p_0\phi/\rho^* + S - k^2/\phi, \tag{8.10}$$

where S is a constant.

9. Surface disturbance of a stream by pressures: small depth

A number of applications of the theory in §8 concerned with the disturbance of a stream of small depth by bodies of specified shape have been given by Caulk (1976) and by Naghdi & Rubin (1981 *a*, *b*, 1982, 1984). Recently, using the same theory, Ertekin *et al.* (1984) have made a detailed numerical study of wave propagation due to a particular surface pressure disturbance and have compared their work with experimental data. Here we study problems similar to those in §5 for a stream of small depth using the theory of §8.

Consider a steady-state stream in the x-direction ($x_1 = x$) and a constant band of pressure $\hat{p} = p_1$ on the stream between $x = x_1$, $x = x_2$, the remaining surface in the regions $x < x_1$, $x > x_2$ having atmospheric pressure $\hat{p} = p_0$. We divide the fluid into three regions and make use of (8.10) with zero surface tension, setting $v_1 = v$. Thus: in region I ($x < x_1$),

$$\begin{aligned} \phi v &= k_1, \\ \tfrac{1}{3}k_1^2(\phi')^2 &= k_1^2 - 2S_1\phi + 2R_1\phi^2 - g\phi^3, \\ p/\rho^* &= p_0\phi/\rho^* + S_1 - k_1^2/\phi; \end{aligned} \tag{9.1}$$

in region II ($x_1 \leqslant x \leqslant x_2$),

$$\begin{aligned} \phi v &= k_2, \\ \tfrac{1}{3}k_2^2(\phi')^2 &= k_2^2 - 2S_2\phi + 2R_2\phi^2 - g\phi^3, \\ p/\rho^* &= p_1\phi/\rho^* + S_2 - k_2^2/\phi; \end{aligned} \tag{9.2}$$

in region III ($x_2 < x$),

$$\begin{aligned} \phi v &= k_3, \\ \tfrac{1}{3}k_3^2(\phi')^2 &= k_3^2 - 2S_3\phi + 2R_3\phi^2 - g\phi^3, \\ p/\rho^* &= p_0\phi/\rho^* + S_3 - k_3^2/\phi. \end{aligned} \tag{9.3}$$

At $x = x_1, x_2$ the wave heights are h_1, h_2, respectively, and at $x \to -\infty$ the stream is of depth h and moving with constant speed c. We seek a solution of the flow problem in which ϕ, ϕ', p and mass flow are continuous at $x = x_1, x_2$ so that, from (9.1)–(9.3), it follows that

$$\left.\begin{aligned} k_1 = k_2 = k_3 &= k = ch, \\ R_1 = \tfrac{1}{2}k^2/h^2 + gh, \quad S_1 &= k^2/h + \tfrac{1}{2}gh^2, \\ S_2 - k^2/h_1 + p_1h_1/\rho^* &= S_1 - k^2/h_1 + p_0h_1/\rho^*, \\ R_2 - S_2/h_1 &= R_1 - S_1/h_1, \\ S_3 - k^2/h_2 + p_0h_2/\rho^* &= S_2 - k^2/h_2 + p_1h_2/\rho^*, \\ R_3 - S_3/h_2 &= R_2 - S_2/h_2. \end{aligned}\right\} \tag{9.4}$$

Thus: in region I,

$$\begin{aligned} \tfrac{1}{3}k^2(\phi')^2 &= (h-\phi)^2(k^2/h^2 - g\phi), \\ p/\rho^* &= p_0\phi/\rho^* + \tfrac{1}{2}gh^2 + k^2/h - k^2/\phi; \end{aligned} \tag{9.5}$$

in region II,

$$\begin{aligned} \tfrac{1}{3}k^2(\phi')^2 &= (h-\phi)^2(k^2/h^2 - g\phi) + 2\mu gh\phi(h_1 - \phi), \\ p/\rho^* &= p_1\phi/\rho^* - \mu ghh_1 + \tfrac{1}{2}gh^2 + k^2/h - k^2/\phi; \end{aligned} \tag{9.6}$$

in region III,

$$\begin{aligned} \tfrac{1}{3}k^2(\phi')^2 &= (h-\phi)^2(k^2/h^2 - g\phi) + 2\mu gh(h_1 - h_2)\,\phi, \\ p/\rho^* &= p_0\phi/\rho^* - \mu gh\,(h_1 - h_2) + \tfrac{1}{2}gh^2 + k^2/h - k^2/\phi, \end{aligned} \tag{9.7}$$

where $p_1 - p_0 = \rho^* gh\mu$. If $h_1 > h_2$, then (9.7) only yields a satisfactory wave form in region III if $F^2 = k^2/(gh^3) = c^2/(gh) < 1$. In this case (9.5) shows that there is motion in region I only if

$$\phi = h, \quad p/\rho^* = p_0h/\rho^* + \tfrac{1}{2}gh^2. \tag{9.8}$$

Then, in region II

$$h_1 = h, \quad \tfrac{1}{3}k^2(\phi')^2 = g(h-\phi)\{\phi^2-\phi h(1+F^2-2\mu)+F^2h^2\}, \tag{9.9}$$

and in region III

$$\tfrac{1}{3}k^2(\phi')^2 = g(h-\phi)^2(F^2h-\phi)+2g\mu h(h-h_2)\,\phi. \tag{9.10}$$

From (9.10) it can be shown that if

$$64\mu(1-h_2/h) < \{F^2+2-(F^4+8F^2)^{\frac{1}{2}}\}\{4-F^2-(F^4+8F^2)^{\frac{1}{2}}\}^2, \tag{9.11}$$

then periodic wave motion is possible in region III. This motion degenerates into a flow in which the height decreases continuously from $\phi = h_2$ to

$$\phi = \tfrac{1}{4}h\{F^2+(F^4+8F^2)^{\frac{1}{2}}\} < h_2 < h$$

if

$$64\mu(1-h_2/h) = \{F^2+2-(F^4+8F^2)^{\frac{1}{2}}\}\{4-F^2-(F^4+8F^2)^{\frac{1}{2}}\}^2.$$

For other values of μ no satisfactory motion is possible. Consider further the special case when the region II becomes small and the force p_1 becomes large such that an isolated force acts on the surface at the point $x = x_1$, $\phi = h$. Let $h_2 = h(1-\epsilon)$ where ϵ is small and assume that μ becomes large in such a way that $\mu\epsilon$ tends to a finite limit. The isolated force has vertical and horizontal components N, L respectively at $x = x_1$ given by

$$N = \int_{x_1}^{x_2} p_1\,\mathrm{d}x \to \rho^* gh\mu(x_2-x_1), \quad L = -\int_{h}^{h_2} p_1\,\mathrm{d}\phi = \rho^* gh^2\mu\epsilon \tag{9.12}$$

when x_2-x_1 and hence ϵ are small, and from (9.9),

$$x_2-x_1 \to (\tfrac{2}{3})^{\frac{1}{2}}\,hF\epsilon/(\mu\epsilon)^{\frac{1}{2}}. \tag{9.13}$$

Hence

$$(\mu\epsilon)^{\frac{1}{2}} \to (\tfrac{3}{2})^{\frac{1}{2}}\,N/(\rho^* gh^2F),$$

$$L \to 3N^2/\{2\rho^* gh^2F^2\}. \tag{9.14}$$

If, in addition, $\mu\epsilon$ and hence N and L are small, then L may be neglected compared with N and, from (9.7), the surface elevation in region III is given approximately by

$$\phi = h-(3N/\rho^* c^2\alpha)\sin(\alpha x/h), \tag{9.15}$$

where

$$\alpha = \{3(1-F^2)\}^{\frac{1}{2}}/F. \tag{9.16}$$

For this special situation, and for streams with small depths, the solution may be compared with that given by Lamb (1932, §245) which was obtained with the help of the three-dimensional linear theory. Lamb shows that upstream of the isolated force there is a disturbance of the surface which dies away rapidly as we move upstream of the isolated force. Downstream there is a similar disturbance together with an harmonic wave of length $2\pi h/\bar{\alpha}$ and maximum and minimum wave heights

$$\pm\frac{2N\bar{\alpha}}{\rho^* c^2\{\bar{\alpha}^2-(1-F^2)/F^4\}}, \tag{9.17}$$

above the undisturbed stream where $\tanh\bar{\alpha} = F^2\bar{\alpha}$. Since $F^2 < 1$ comparison between the solution given by Lamb and the special case of the present (nonlinear) theory for small values of N and h appears only to be relevant for values of F^2 in the neighbourhood of 1. Then $\bar{\alpha} \approx \alpha$

and the results from Lamb and the present theory agree as far as the wave-like part of the motion is concerned. For small values of F^2 Lamb's solution should be compared with the isolated force solution based on the theory of §4 and given in §5.

When $F^2 > 1$ the character of the flows in the three regions is different from that already discussed. We now assume that in region III downstream of the constant pressure p_1, the stream is of constant depth and moves with speed c as $x \to \infty$. Adopting (9.1)–(9.3) for the three regions the continuity conditions at $x = x_1$, x_2 and the conditions at $x \to \infty$ now yield

$$\left.\begin{aligned} k_1 = k_2 = k_3 = k = ch^2, \\ S_1 - k^2/h_1 + p_0 h_1/\rho^* = S_2 - k^2/h_1 + p_1 h_1/\rho^*, \\ R_1 - S_1/h_1 = R_2 - S_2/h_1, \\ S_2 - k^2/h_2 + p_1 h_2/\rho^* = S_3 - k^2/h_2 + p_0 h_2/\rho^*, \\ R_2 - S_2/h_2 = R_3 - S_3/h_2, \\ R_3 = \tfrac{1}{2}k^2/h^2 + gh, \quad S_3 = k^2/h + \tfrac{1}{2}gh^2. \end{aligned}\right\} \tag{9.18}$$

Thus: in region III:

$$\tfrac{1}{3}F^2(\phi')^2 = (1-\phi/h)^2\,(F^2-\phi/h), \quad F^2 = c^2/(gh) > 1; \tag{9.19}$$

in region II:

$$\tfrac{1}{3}F^2(\phi')^2 = (1-\phi/h)^2\,(F^2-\phi/h) + 2\mu(\phi/h)\,(h_2/h-\phi/h); \tag{9.20}$$

in region I:

$$\tfrac{1}{3}F^2(\phi')^2 = (1-\phi/h)^2\,(F^2-\phi/h) - 2\mu(h_1-h_2)\,\phi/h^2. \tag{9.21}$$

Here $F^2 > 1$ although h is now the downstream instead of the upstream depth. From (9.21) we see that wave motion is possible if

$$64\mu(h_1-h_2)/h < \{(F^4+8F^2)^{\frac{1}{2}}+F^2-4\}^2\{(F^4+8F^2)^{\frac{1}{2}}-F^2-2\}. \tag{9.22}$$

This type of wave motion ahead of the pressure disturbance is similar to that discussed numerically by Ertekin *et al.* (1984) for a different pressure disturbance and for unsteady flow. The particular case of an isolated force may be considered as a special case of the foregoing discussion.

Again, as in the problem of a pressure distribution on a stream of infinite depth, other motions which involve waves both upstream and downstream of applied pressure are also possible.

10. Gliding on the surface of a stream: small depth

Here we consider the problem of gliding of a plate on a stream of small depth, which is similar to the problem discussed in §6 for infinite depth. The nonlinear steady-state solution to the problem of the transition to planing of a two-dimensional boat has been solved by Naghdi & Rubin (1981*a*) on the basis of the theory given in §8 for $F < 1$, where $F^2 = c^2/(gh)$ and h is the upstream depth of the fluid and c its speed. Here we limit attention to the case $F > 1$.

The height of the trailing edge of the plate which is at the point $x = x_2$ is h_2, and the leading edge $x = x_1$ is at a height $h_1 = h_2 + b\tan\alpha$, where $b = l\cos\alpha$, l is the length of the plate and α its inclination to the horizontal. We divide the fluid into three regions as in §9. In regions I and III the surface pressure, $\hat{p}$, has the prescribed constant value p_0, whereas in region II the surface has prescribed shape which here is $\phi = h_2 + (b-x)\tan\alpha$ with an unknown $\hat{p}$ on this surface. In region I,

$$\phi v = k_1, \quad \tfrac{1}{3}k_1^2(\phi')^2 = k_1^2 - 2S_1\phi + 2R_1\phi^2 - g\phi^3, \quad p/\rho^* = p_0\phi/\rho^* + S_1 - k_1^2/\phi; \tag{10.1 a–c}$$

in region II,

$$\left.\begin{aligned} \phi v = k_2, \quad \phi &= h_2 + (b-x)\tan\alpha, \\ p/\rho^* &= p_0\phi/\rho^* + R_2\phi - \tfrac{1}{2}g\phi^2 - (k_2^2/2\phi)(1+\tfrac{1}{3}\tan^2\alpha), \\ \hat{p}/\rho^* &= p_0/\rho^* + R_2 - g\phi - (k_2^2/2\phi^2)(1-\tfrac{1}{3}\tan^2\alpha), \end{aligned}\right\} \tag{10.2}$$

in region III,

$$\left.\begin{aligned} \phi v &= k_3, \\ \tfrac{1}{3}k_3^2(\phi')^2 &= k_3^2 - 2S_3\phi + 2R_3\phi^2 - g\phi^3, \\ p/\rho^* &= p_0\phi/\rho^* + S_3 - k_3^2/\phi. \end{aligned}\right\} \tag{10.3}$$

For region III, at $x\to\infty$ we assume that the stream has speed c and that $\phi = h$, $\phi' = \phi'' = 0$, so that in region III,

$$\left.\begin{aligned} \phi v &= k_3 = ch = k, \\ \tfrac{1}{3}k^2(\phi')^2 &= (\phi-h)^2(k^2/h^2 - g\phi), \\ p/\rho^* &= p_0\phi/\rho^* + S_3 - k^2/\phi. \end{aligned}\right\} \tag{10.4}$$

Continuity of ϕ, ϕ', ϕv, p at $x = x_1, x_2$ yields

$$k_1 = k_2 = k, \tag{10.5a}$$

$$\tfrac{1}{3}k^2\tan^2\alpha = (h-h_2)^2(k^2/h^2 - gh_2), \tag{10.5b}$$

$$\frac{k^2}{h} + \tfrac{1}{2}gh^2 = R_2h_2 + k^2/2h_2 - \tfrac{1}{2}gh_2^2 - (k^2\tan^2\alpha)/6h_2, \tag{10.5c}$$

$$S_1 = R_2h_1 + k^2/2h_1 - \tfrac{1}{2}gh_1^2 - (k^2\tan^2\alpha)/6h_1, \tag{10.5d}$$

$$\tfrac{1}{3}k^2\tan^2\alpha = k^2 - 2S_1h_1 + 2R_1h_1^2 - gh_1^3, \tag{10.5e}$$

$$h_1 = h_2 + b\tan\alpha. \tag{10.5f}$$

Hence

$$R_1 = R_2 = \tfrac{1}{2}k^2/h^2 + gh \tag{10.6}$$

and (10.5 b) yields a possible value h_2, if

$$\tan^2\alpha \leqslant 4(F^2-1)^3/(9F^2), \quad F^2 > h_2/h > 1, \quad F^2 = c^2/(gh) > 1. \tag{10.7}$$

We confine further discussion to the case when the angle of planing is small. Then, it follows from (10.5), that

$$\begin{gathered} h_2 = h[1 + F\{3(F^2-1)\}^{-\frac{1}{2}}\tan\alpha], \quad S_1 = k^2/h + \tfrac{1}{2}gh^2 + \tfrac{1}{2}gh^2\gamma\tan^2\alpha, \\ \gamma = (F^2-1)(b/h)[b/h + 2F\{3(F^2-1)\}^{-\frac{1}{2}}], \end{gathered} \tag{10.8}$$

approximately. Equation (10.1 b) in region I becomes

$$\tfrac{1}{3}F^2(\phi')^2 = (1-\phi/h)^2(F^2-\phi/h) - \gamma(\phi/h)\tan^2\alpha. \tag{10.9}$$

This represents a periodic wave with maximum and minimum heights

$$hF^2\{1-\gamma(F^2-1)^{-2}\tan^2\alpha\} \quad\text{and}\quad h\{1+\gamma^{\frac{1}{2}}(F^2-1)^{-\frac{1}{2}}\tan\alpha\}, \tag{10.10}$$

respectively. From (10.2) we obtain L, the net total lift normal to the plate, in the form

$$L = \int_{h_2}^{h_1} \frac{(\hat{p}-p_0)\,\mathrm{d}\phi}{\sin\alpha} = \tfrac{1}{2}\rho^* lc^2\left(\frac{F^2-1}{F^2}\right)\left[\frac{b}{h}+\frac{2F}{\{3(F^2-1)\}^{\frac{1}{2}}}\right]\tan\alpha \tag{10.11}$$

approximately.

The foregoing model for the gliding of a plate when the downstream Froude number $F > 1$ is likely to be unsatisfactory when gravity effects are small compared with inertia effects, i.e. when F is large. Then the maximum wave height (10.23) ahead of the plate becomes very large and the flow is likely to break up into a spray at the leading edge of the plate. It is then appropriate to use the theory of Green (1935, 1936*a*, *b*, 1938).

Gliding which involves waves both upstream and downstream of the plate is also possible but is not discussed here.

11. Nonlinear stern waves: small depth

The problem of nonlinear stern waves discussed in §7 has not been discussed with a three-dimensional theory when the depth is finite, so we consider this problem here with the use of the small depth theory of §8. Consider two regions in which an incompressible inviscid fluid is bounded below by a straight bottom and flow is two-dimensional. In region I, $-\infty < x \leqslant 0$, the fluid is bounded above by a fixed boundary $\phi = h$, the stream speed is c and the pressure at the fluid boundary is p_1 In region II, $0 \leqslant x < \infty$, the upper surface is free and at constant pressure p_0, and leaves the upper fixed surface at $x = 0$ smoothly. The upper boundary in region I represents the bottom of a flat-bottomed boat of draft H so that

$$p_1 - p_0 = \rho^* gH. \tag{11.1}$$

Then, from §8, for steady motion: in region I,

$$\phi = h, \quad p = p_1 h + \tfrac{1}{2}\rho^* gh^2; \tag{11.2}$$

in region II,

$$\phi v = k$$
$$\tfrac{1}{3}k^2(\phi')^2 = k^2 - 2S\phi + 2R\phi^2 - g\phi^3, \tag{11.3}$$
$$p/\rho^* = p_0\phi/\rho^* + S - k^2/\phi.$$

At $x = 0$, ϕv, ϕ', p are continuous so that

$$k = ch, \quad S = k^2/h + \tfrac{1}{2}gh^2 + ghH, \quad R = \tfrac{1}{2}k^2/h^2 + g(h+H) \tag{11.4}$$

and in region II

$$\tfrac{1}{3}k^2(\phi')^2 = (\phi-h)^2(k^2/h^2 - g\phi) + 2gH\phi(\phi-h),$$

or

$$\tfrac{1}{3}F^2(\beta')^2 = \beta^2(F^2-1-\beta) + 2m\beta(1+\beta), \tag{11.5}$$
$$m = H/h, \quad F^2 = c^2/(gh), \quad \phi = h(1+\beta), \quad x = h\bar{x},$$

and a prime denotes differentiation with respect to $\bar{x}$. Wave motion is possible in region II with minimum and maximum heights

$$\beta = 0, \quad \beta = \beta_1, \quad 2\beta_1 = F^2 - 1 + 2m + \{(F^2-1+2m)^2 + 8m\}^{\frac{1}{2}}. \tag{11.6}$$

The wavelength, λ, is then given by

$$\frac{\lambda}{h} = \frac{2F}{\sqrt{3}} \int_0^{\beta_1} \frac{\mathrm{d}\beta}{\{\beta(\beta_1-\beta)(\beta_2+\beta)\}^{\frac{1}{2}}} = \frac{4\bar{k}FK(\bar{k})}{(3\beta_1)^{\frac{1}{2}}},$$

$$\beta_2 = 2m/\beta_1, \quad \bar{k}^2 = \beta_1^2/(\beta_1^2+2m), \tag{11.7}$$

where $K(\bar{k})$ is the elliptic integral defined by

$$K(\bar{k}) = \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}\theta}{(1-\bar{k}^2\sin^2\theta)^{\frac{1}{2}}}. \tag{11.8}$$

It is now straightforward to compute maximum wave height, wavelength and steepness of the waves in region II for ranging values of F and m.

PART D. FLUIDS OF FINITE DEPTH

12. Introduction

In parts B, C we established theories for wave propagation on inviscid incompressible fluids of constant density ρ^* which had either small or infinite depths. Here we obtain a theory for such fluids which have finite depths, based on the general theory of §2. The surface, $\mathscr{S}$, is again a plane surface $\zeta = x_3 = 0$ and we use the notation of (4.1). The fluid is subject to constant gravity g in the $-\boldsymbol{e}_3$ direction and is bounded by the fixed surface $x_3 = -h$, where h is a constant, and the surface of the stream is again given by (4.2), at which there is a pressure $\hat{p} = \hat{p}(x_1, x_2, t)$ and a constant surface tension T. The pressure at the bottom surface is $\bar{p}$. From §2 we select the theory with three directors corresponding in the appendix to the choice of weighting functions, $\lambda_1(\zeta) = \cosh\{a(\zeta+h)\}$, $\lambda_2(\zeta) = \sinh\{a(\zeta+h)\}$, $\lambda_3(\zeta) = \zeta$, with a being constant and we choose

$$\bar{\boldsymbol{d}}_0 = x_\alpha \boldsymbol{e}_\alpha + d\boldsymbol{e}_3, \quad \bar{\boldsymbol{d}}_1 = \bar{\boldsymbol{d}}_2 = \boldsymbol{0}, \quad \bar{\boldsymbol{d}}_3 = \boldsymbol{e}_3, \tag{12.1}$$

where d is a constant. Also, we specify the velocity $\boldsymbol{v}$ to be constant in a direction parallel to the $\mathscr{S}$ plane and the velocity $\boldsymbol{w}_3$ to be zero so that

$$\boldsymbol{v} = c_\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{w}_3 = \boldsymbol{0}, \tag{12.2}$$

where c_α are constants. The director velocities have components

$$\boldsymbol{w}_1 = w_{1i}\boldsymbol{e}_i, \quad \boldsymbol{w}_2 = w_{2i}\boldsymbol{e}_i, \tag{12.3}$$

and in the light of (A 19) the incompressibility conditions are

$$w_{1\alpha,\alpha} + aw_{23} = 0, \quad w_{2\alpha,\alpha} + aw_{13} = 0. \tag{12.4}$$

In addition to the incompressibility constraint (12.4), and the constraint (12.2) on $\boldsymbol{v}$ we adopt a further simplification in the theory by imposing the extra constraint

$$w_{2\alpha} = 0. \tag{12.5}$$

Then, in view of (12.2), (12.4) and (12.5), the constitutive equations for an incompressible inviscid fluid are

$$\left.\begin{aligned} \boldsymbol{N}_\alpha = r_{i\alpha}\boldsymbol{e}_i, \quad \boldsymbol{k}_1 = -p_1 a\boldsymbol{e}_3, \quad \boldsymbol{k}_2 = q_\alpha \boldsymbol{e}_\alpha - pa\boldsymbol{e}_3, \\ \boldsymbol{M}_{1\alpha} = -p\boldsymbol{e}_\alpha, \quad \boldsymbol{M}_{2\alpha} = -q_{\beta\alpha}\boldsymbol{e}_\beta - p_1\boldsymbol{e}_\alpha, \quad \boldsymbol{M}_{3\alpha} = s_{i\alpha}\boldsymbol{e}_i, \quad \boldsymbol{k}_3 = s_i\boldsymbol{e}_i, \end{aligned}\right\} \tag{12.6}$$

where $r_{i\alpha}$, q_α, $q_{\alpha\beta}$, p, p_1, $s_{i\alpha}$, s_i are functions of x_α, t. With the help of (12.1) these satisfy (2.12) if

$$r_{3\alpha} = s_\alpha, \quad r_{\alpha\beta} = r_{\beta\alpha}. \tag{12.7}$$

In view of (A 11) and (A 17) we choose the following values for y_{MN}, $\boldsymbol{v}_{MN}$, $\boldsymbol{f}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$:

$$\left.\begin{aligned}
\rho y_{00} &= \rho^*\phi = \rho^*(\beta+h), \quad \rho y_{10} = (\rho^*/a)\sinh(a\phi),\\
\rho y_{20} &= (\rho^*/a)\{\cosh(a\phi)-1\}, \quad \rho y_{11} = (\rho^*/2a)\{\sinh(a\phi)\cosh(a\phi)+a\phi\},\\
\rho y_{12} &= (\rho^*/2a)\sinh^2(a\phi),\\
\rho y_{22} &= (\rho^*/2a)\{\sinh(a\phi)\cosh(a\phi)-a\phi\},\\
\rho\boldsymbol{v}_{00} &= \{\rho^* c_\alpha\phi + (\rho^*/a)\,w_{1\alpha}\sinh(a\phi)\}\,\boldsymbol{e}_\alpha\\
\rho\boldsymbol{v}_{10} &= (\rho^*/a)\,[c_\alpha\sinh(a\phi)+\tfrac{1}{2}\{\sinh(a\phi)\cosh(a\phi)+a\phi\}\,w_{1\alpha}\}\,\boldsymbol{e}_\alpha\\
&\quad +\tfrac{1}{2}\rho^*\{\sinh(a\phi)\cosh(a\phi)-a\phi\}\,w_{23}\,\boldsymbol{e}_3,\\
\rho\boldsymbol{v}_{20} &= (\rho^*/a)\,[c_\alpha\{\cosh(a\phi)-1\}+\tfrac{1}{2}\sinh^2(a\phi)\,w_{1\alpha}]\,\boldsymbol{e}_\alpha+\tfrac{1}{2}\rho^*\sinh^2(a\phi)\,w_{23}\,\boldsymbol{e}_3,
\end{aligned}\right\} \tag{12.8}$$

$$\left.\begin{aligned}
\rho\boldsymbol{v}_{11} &= (\rho^*/a)\,[\tfrac{1}{2}c_\alpha\{\sinh(a\phi)\cosh(a\phi)+a\phi\}\\
&\quad +\{\tfrac{1}{3}\sinh^3(a\phi)+\sinh(a\phi)\}\,w_{1\alpha}]\,\boldsymbol{e}_\alpha+\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{23}\,\boldsymbol{e}_3,\\
\rho\boldsymbol{v}_{01} &= (\rho^*/a)\,[c_\alpha\sinh(a\phi)+\tfrac{1}{2}\{\sinh(a\phi)\cosh(a\phi)+a\phi\}\,w_{1\alpha}]\,\boldsymbol{e}_\alpha,\\
\rho\boldsymbol{v}_{02} &= (\rho^*/a)\,[c_\alpha\{\cosh(a\phi)-1\}+\tfrac{1}{2}\sinh^2(a\phi)\,w_{1\alpha}]\,\boldsymbol{e}_\alpha,\\
\rho\boldsymbol{v}_{12} &= (\rho^*/a)\,[\tfrac{1}{2}c_\alpha\sinh^2(a\phi)+\tfrac{1}{3}\{\cosh^3(a\phi)-1\}\,w_{1\alpha}]\,\boldsymbol{e}_\alpha\\
&\quad +\rho^*\{\tfrac{1}{3}\cosh^3(a\phi)-\cosh(a\phi)+\tfrac{2}{3}\}\,w_{23}\,\boldsymbol{e}_3,\\
\rho\boldsymbol{v}_{22} &= (\rho^*/a)\,[\tfrac{1}{2}c_\alpha\{\sinh(a\phi)\cosh(a\phi)-a\phi\}+\tfrac{1}{3}\sinh^3(a\phi)\,w_{1\alpha}]\,\boldsymbol{e}_\alpha\\
&\quad +\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{23}\,\boldsymbol{e}_3,\\
\rho\boldsymbol{v}_{21} &= (\rho^*/a)\,[\tfrac{1}{2}c_\alpha\sinh^2(a\phi)+\tfrac{1}{3}\{\cosh^3(a\phi)-1\}\,w_{1\alpha}]\,\boldsymbol{e}_\alpha\\
&\quad +\tfrac{1}{3}\rho^*\{\cosh^3(a\phi)-1\}\,w_{23}\,\boldsymbol{e}_3,
\end{aligned}\right\} \tag{12.9}$$

and

$$\left.\begin{aligned}
\rho\boldsymbol{f} &= (\hat{p}-q)\,\phi_{,\alpha}\boldsymbol{e}_\alpha-(\hat{p}-q-\bar{p})\,\boldsymbol{e}_3-\rho^* g\phi\boldsymbol{e}_3,\\
\rho\boldsymbol{l}_1 &= (\hat{p}-q)\,\phi_{,\alpha}\cosh(a\phi)\,\boldsymbol{e}_\alpha-\{(\hat{p}-q)\cosh(a\phi)-\bar{p}\}\,\boldsymbol{e}_3-\rho^*(g/a)\sinh(a\phi)\,\boldsymbol{e}_3,\\
\rho\boldsymbol{l}_2 &= (\hat{p}-q)\,\phi_{,\alpha}\sinh(a\phi)\,\boldsymbol{e}_\alpha-(\hat{p}-q)\sinh(a\phi)\,\boldsymbol{e}_3-\rho^*(g/a)\{\cosh(a\phi)-1\}\,\boldsymbol{e}_3,
\end{aligned}\right\} \tag{12.10}$$

where $\phi = \beta+h$. As previously q is given by (4.10) or by (4.10) with β replaced by ϕ.

With the help of (12.4)–(12.10) the field equations (2.9), (2.10) and (2.11) reduce to

$$\partial\phi/\partial t+\{c_\alpha+w_{1\alpha}\cosh(a\phi)\}\,\partial\phi/\partial x_\alpha-w_{23}\sinh(a\phi)=0, \tag{12.11}$$

$$\begin{aligned}(\rho^*/2a)\{\sinh(a\phi)\cosh(a\phi)+a\phi\}\,\partial w_{1\alpha}/\partial t+(\rho^*/a)\,[\tfrac{1}{2}c_\beta\{\sinh(a\phi)\cosh(a\phi)+a\phi\}\\ +\{\tfrac{1}{3}\sinh^3(a\phi)+\sinh(a\phi)\}\,w_{1\beta}]\,\partial w_{1\alpha}/\partial x_\beta+\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{23}\,w_{1\alpha}\\ =-\partial p/\partial x_\alpha+(\hat{p}-q)\cosh(a\phi)\,\partial\phi/\partial x_\alpha,\end{aligned} \tag{12.12}$$

$$\begin{aligned}(\rho^*/2a)\{\sinh(a\phi)\cosh(a\phi)-a\phi\}\,\partial w_{23}/\partial t+(\rho^*/a)\,[\tfrac{1}{2}c_\alpha\{\sinh(a\phi)\cosh(a\phi)-a\phi\}\\ +\tfrac{1}{3}\sinh^3(a\phi)\,w_{1\alpha}]\,\partial w_{23}/\partial x_\alpha+\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{23}^2\\ =ap-(\hat{p}-q)\sinh(a\phi)-\rho^*(g/a)\{\cosh(a\phi)-1\}.\end{aligned} \tag{12.13}$$

The equations involving the constraint response functions p_1, $r_{i\alpha}$, q_α, $q_{\alpha\beta}$, $s_{i\alpha}$, s_i and the pressure $\bar{p}$ on the bed of the stream are omitted, since the values of these constraint response functions are not required in the further development of the theory.

As in the theory of §4, it is also instructive here to examine the linearized form of (12.4) and (12.11)–(12.13). With $\phi=h+\beta$ we have

$$\left.\begin{aligned}&\partial w_{1\alpha}/\partial x_\alpha+aw_{23}=0,\quad \partial\beta/\partial t+c_\alpha\,\partial\beta/\partial x_\alpha-w_{23}\sinh(ah)=0,\\ &(\rho^*/2a)\{\sinh(ah)\cosh(ah)+ah\}\,(\partial w_{1\alpha}/\partial t+c_\beta\,\partial w_{1\alpha}/\partial x_\beta)=-\partial p/\partial x_\alpha,\\ &(\rho^*/2a)\{\sinh(ah)\cosh(ah)-ah\}\,(\partial w_{23}/\partial t+c_\beta\,\partial w_{23}/\partial x_\beta)\\ &\quad=ap-(\rho^*g/a)\{\cosh(ah)-1+a\beta\sinh(ah)\}-(\hat{p}-T\partial^2\beta/\partial x_\alpha\partial x_\alpha)\sinh(ah).\end{aligned}\right\} \tag{12.14}$$

One example of the use of these equations is for wave propagation in the x_1-direction when $c_\beta=0$ of the form $\beta=L\sin m(x-ct)$, when $\hat{p}$ is constant, or equivalently, standing waves $\beta=L\sin mx$ on a uniform stream $c_1=c$. Either is a possible solution of (12.14) provided that

$$\rho^*c^2\{(a^2+m^2)\cosh(ah)+(a^2-m^2)\,ah\}=2(\rho^*g+Tm^2)\,a\sinh(ah). \tag{12.15}$$

We again choose $a=m$ so that (12.15) reduces to

$$c^2=(g/m)\{1+Tm^2/(\rho^*g)\}\tanh(mh),$$

which is identical with that found from the linearized three-dimensional theory (Whitham 1974, §12.1).

Returning to (12.4) and (12.11)–(12.13) we consider travelling waves or, equivalently steady motion waves on a stream moving with constant speed c in the x_1-direction, the surface pressure $\hat{p}$ being equal to p_0, a constant, and zero surface tension. Then $c_1=c$, $c_2=0$ and ϕ, w_{11}, w_{23} and p are functions of x_1. Using a prime to denote differentiation with respect to x, it follows from (12.4) and (12.11) that

$$\left.\begin{aligned}w_{11}&=c(A-a\beta)/\sinh(a\phi),\\ w_{23}&=c\{(A-a\beta)\cosh(a\phi)+\sinh(a\phi)\}\,\phi'/\sinh^2(a\phi),\end{aligned}\right\} \tag{12.16}$$

where A is a constant. One component of the equations of motion is satisfied identically and, with the help of (12.16), the other component in (12.12) yields

$$\begin{aligned}p/\rho^*=(p_0/a\rho^*)\sinh(a\phi)+Bg/a^2-(c/2a)\{\sinh(a\phi)\cosh(a\phi)+a\phi+2(A-a\beta)\}\,w_{11}\\ +(c^2/a)\{(A-a\beta)\cosh(a\phi)+\sinh(a\phi)\},\end{aligned} \tag{12.17}$$

where B is a constant. With the help of (12.16) again (12.13) gives

$$p/\rho^* = (p_0/a\rho^*)\{\sinh(a\phi)\} + (g/a^2)\{\cosh(a\phi)-1\}$$
$$+(c/12a^3w'_{11})[(w'_{11})^2\{-2(A-a\beta)\sinh^2(a\phi)-3\sinh(a\phi)\cosh(a\phi)+3a\phi\}]'. \quad (12.18)$$

By eliminating p between (12.17) and (12.18) and performing one integration, we have

$$\tfrac{1}{12}\Gamma(\phi')^2[\{(A-a\beta)\cosh(a\phi)+\sinh(a\phi)\}/\sinh^2(a\phi)]^2$$
$$\times\{3\sinh(a\phi)\cosh(a\phi)-3a\phi+2(A-a\beta)\sinh^2(a\phi)\}$$
$$=-(A-a\beta)\{B+1-\cosh(a\phi)\}/\sinh(a\phi)+\tfrac{1}{2}(A-a\beta)^2$$
$$+\Gamma[(A-a\beta)^2\{2A-a(\beta-h)\}/4\sinh^2(a\phi)-3(A-a\beta)^2\cosh(a\phi)/4\sinh(a\phi)$$
$$-\tfrac{1}{6}(A-a\beta)^3-(A-a\beta)]+E, \quad (12.19)$$

where E is a constant and

$$\Gamma = ac^2/g. \quad (12.20)$$

The properties of nonlinear progressive waves of finite amplitude on water of any depth, or equivalently, the properties of standing waves on a uniform stream may be discussed with the aid of (12.6)–(12.19). Suppose now that the constants A, B and E are chosen so that when $\beta = 0$ ($\phi = h$), $\beta' = 0$, $\beta'' = 0$, $w_{11} = 0$. Then

$$A = 0,\ E = 0,\ B = \cosh(ah)-1-\Gamma\sinh(ah) \quad (12.21)$$

and (12.19) reduces to

$$\tfrac{1}{12}\Gamma(\phi')^2[\{a\beta\cosh(a\phi)-\sinh(a\phi)\}/\sinh^2(a\phi)]^2\{3\sinh(a\phi)\cosh(a\phi)-3a\phi-2a\beta\sinh^2(a\phi)\}$$
$$= a\beta\{\cosh(ah)-\cosh(a\phi)\}/\sinh(a\phi)+\tfrac{1}{2}a^2\beta^2+\Gamma\{-a^3\beta^2(\beta-h)/4\sinh^2(a\phi)$$
$$-3a^2\beta^2\cosh(a\phi)/4\sinh(a\phi)-a\beta\sinh(ah)/\sinh(a\phi)+\tfrac{1}{6}a^3\beta^3+a\beta\}. \quad (12.22)$$

The right-hand side of (12.22) has a double zero at $a\beta = 0$, and is positive in the neighbourhood of this double zero if

$$\Gamma > 2\sinh^2(ah)/\{\sinh(ah)\cosh(ah)+ah\}. \quad (12.23)$$

If the right-hand side of (12.22) has a positive simple zero which is less than the single zero of

$$\tanh a(\beta+h) = a\beta \quad (12.24)$$

then one solution of (12.22) represents a solitary wave. Two limiting cases of (12.22) are of interest. When the fluid has infinite depth, $ah\to\infty$ and (12.22) reduces to the simpler equation (4.29). On the other hand, when the depth is small compared with a characteristic length then $ah\to 0$, $a\beta\to 0$ and (12.22) becomes

$$\tfrac{1}{3}c^2h^2(\beta')^2 = \beta^2(c^2-gh-g\beta), \quad (c^2 > gh) \quad (12.25)$$

which is the same equation as that derived by Green *et al.* (1974) using a direct theory with one director, as in §8.

13. Surface disturbance of a stream by pressures: finite depth

We consider the same problem as that discussed in §§5 and 9 but now using the theory of §12. There is a steady-state stream in the x-direction ($x_1 = x$) on which is imposed the surface pressures specified in (5.1). As in previous sections we divide the stream into three regions and make use of (12.16), (12.17) and (12.19) in each of the regions $x < x_1$, $x_1 \leqslant x \leqslant x_2$, $x_2 < x$. In region I we have these equations with constants A_1, B_1, E_1; in region II the same equations with constants A_3, B_3, E_3 and in region III the constants A_2, B_2, E_2 with p_0 replaced by p_1 in (12.17). At $x = x_1, x_2$ the wave heights are $\phi = h_1, h_2$ respectively, or $\beta = \beta_1, \beta_2$ where $\phi = h+\beta$ and at $x \to -\infty$ the stream is of depth h and moving with speed c, with $\phi' = \phi'' = 0$. Hence

$$A_1 = 0, \quad E_1 = 0, \quad B_1 = \cosh(ah) - 1 - \Gamma \sinh(ah) \tag{13.1}$$

and the flow in region I is then governed by an equation of the form (12.22). The type of flow in region I then depends on the values of Γ which may be

$$\Gamma \gtrless 2\sinh^2(ah)/\{\sinh(ah)\cosh(ah)+ah\}. \tag{13.2}$$

We seek a solution of the flow problem in which ϕ, ϕ', p and mass flow are continuous at $x = x_1, x_2$. This yields the values

$$\begin{gathered} A_2 = 0, \quad A_3 = 0, \\ B_2 = \cosh(ah) - 1 - \Gamma\sinh(ah) - \mu\sinh(ah_1), \\ B_3 = \cosh(ah) - 1 - \Gamma\sinh(ah) + \mu\{\sinh(ah_2) - \sinh(ah_1)\}, \\ E_2 = \mu a\beta_1, \quad E_3 = \mu a(\beta_1-\beta_2), \quad p_1 - p_0 = \rho^*\mu g/a. \end{gathered} \tag{13.3}$$

Then, in region II:

$$\begin{aligned} \tfrac{1}{12}\Gamma(\phi')^2 &[\{a\beta\cosh(a\phi)-\sinh(a\phi)\}/\sinh^2(a\phi)]^2\{3\sinh(a\phi)\cosh(a\phi)-3a\phi-2a\beta\sinh^2(a\phi)\} \\ &= a\beta\{\cosh(ah)-\cosh(a\phi)\}/\sinh(a\phi)+\tfrac{1}{2}a^2\beta^2+\Gamma\{-a^3\beta^2(\beta-h)/4\sinh^2(a\phi) \\ &\qquad -3a^2\beta^2\cosh(a\phi)/4\sinh(a\phi)-a\beta\sinh(ah)/\sinh(a\phi) \\ &\qquad +\tfrac{1}{6}a^3\beta^3+a\beta\}+\mu a\beta_1-\mu a\beta\sinh(ah_1)/\sinh(a\phi) \end{aligned} \tag{13.4}$$

and in region III:

$$\begin{aligned} \tfrac{1}{12}\Gamma(\phi')^2 &[\{a\beta\cosh(a\phi)-\sinh(a\phi)\}/\sinh^2(a\phi)]^2\{3\sinh(a\phi)\cosh(a\phi)-3a\phi-2a\beta\sinh^2(a\phi)\} \\ &= a\beta\{\cosh(ah)-\cosh(a\phi)\}/\sinh(a\phi)+\tfrac{1}{2}a^2\beta^2+\Gamma\{-a^3\beta^2(\beta-h)/4\sinh^2(a\phi) \\ &\qquad -3a^2\beta^2\cosh(a\phi)/4\sinh(a\phi)-a\beta\sinh(ah)/\sinh(a\phi) \\ &\qquad +\tfrac{1}{6}a^3\beta^3+a\beta\}+\mu a(\beta_1-\beta_2)-[\mu a\beta\{\sinh(ah_1)-\sinh(ah_2)\}]/\sinh(a\phi) \end{aligned} \tag{13.5}$$

If $\beta_1 > \beta_2$ and $\Gamma \geqslant 2\sinh^2(ah)/\{\sinh(ah)\cosh(ah)+ah\}$ then no satisfactory wave motion is possible in region III. We restrict Γ so that

$$\Gamma < 2\sinh^2(ah)/\{\sinh(ah)\cosh(ah)+ah\}. \tag{13.6}$$

From (13.1) and the equations for region I, it is then seen that throughout region I

$$\phi = h, \quad \beta = 0, \quad h_1 = h, \quad \beta_1 = 0. \tag{13.7}$$

We limit further development here to the problem of an isolated force acting at the point $x = x_1$. Then $x_2 - x_1 \to 0$ and $\mu \to \infty$, $\beta_2 \to 0$ in such a way that $\mu(x_2-x_1)$ is finite. Then, from (13.4), we see that

$$\begin{gathered} x_2 - x_1 \to \Gamma^{\frac{1}{2}}\bar{H}\{\sinh(ah)\cosh(ah)-ah\}^{\frac{1}{2}}/\{(\mu\bar{H}a)^{\frac{1}{2}}\sinh(ah)\}, \\ \beta_2 = -\bar{H} \to 0. \end{gathered} \tag{13.8}$$

The pressure acts normally to the surface over the interval x_2-x_1 so that, in the limit, the isolated force has vertical and horizontal components N, L respectively at $x = x_1$ given by

$$\begin{aligned} N &\to \rho^* g\mu(x_2-x_1)/a \\ &\to \rho^* c^2(\mu\bar{H}/a\Gamma)^{\frac{1}{2}}\{\sinh(ah)\cosh(ah)-ah\}^{\frac{1}{2}}/\sinh(ah), \\ L &\to \rho^* g\bar{H}/a \to \rho c^2(\mu\bar{H})/\Gamma. \end{aligned} \tag{13.9}$$

If, in addition, $\mu(x_2-x_1)$, and hence N and L are small, then L may be neglected compared with N, and from (13.5), the surface elevation in region III is given, approximately, by

$$\beta = -\frac{2N}{\rho^* c^2}\frac{\Gamma^{\frac{1}{2}}\sinh^2(ah)\sin mx}{\{\sinh(ah)\cosh(ah)-ah\}^{\frac{1}{2}}[2\sinh^2(ah)-\Gamma\{\sinh(ah)\cosh(ah)+ah\}]^{\frac{1}{2}}}, \tag{13.10}$$

where

$$\frac{m^2}{a^2} = \frac{2\sinh^2(ah)-\Gamma\{\sinh(ah)\cosh(ah)+ah\}}{\Gamma\{\sinh(ah)\cosh(ah)-ah\}}. \tag{13.11}$$

These results for small values of N may be compared with those derived by Lamb (1932, §245) using the linear three-dimensional theory of inviscid fluid flow. Lamb shows that the surface elevation consists of two parts. One part dies away rapidly as we move away from the isolated force, whereas the second part consists downstream of a harmonic wave

$$\beta = -\frac{2N\alpha\sin(\alpha x/h)}{\rho^* c^2\{\alpha^2-F^{-2}(F^{-2}-1)\}}, \quad F^2\alpha = \tanh\alpha, \tag{13.12}$$

where $F^2 = c^2/(gh) < 1$. This part of the solution is identical with that given in (13.10) if we choose the coefficient a so that

$$\Gamma = c^2 a/g = \tanh(ah), \quad ah = \alpha \tag{13.13}$$

and, from (13.11) $m = a$. The inequality (13.6) is satisfied.

As in §5 for a fluid with infinite depth we may continue a study of the nonlinear isolated force problem by returning to (13.5), using the value (13.13) for Γ, but we omit details. Likewise we leave aside any further discussion of the problem for values of Γ not satisfying (13.5).

It is clear that we may use the theory of §12 to discuss the remaining problems considered in previous sections when limited to either small or infinite depths, using similar techniques, but again we omit details.

The work of one of us (P.M.N.) was supported by the Fluid Mechanics Program of the U.S. Office of Naval Research under contract N00014-86-K-0057, Work Unit 4322-534, with the University of California, Berkeley.

Appendix

Let ζ^i $(i = 1, 2, 3)$ be a system of curvilinear coordinates in Euclidean three-space and let points in this space be specified by a position vector $\boldsymbol{r}^* = \boldsymbol{r}^*(\zeta^i)$. The base vectors and metric tensors are given by

$$\begin{gathered} \boldsymbol{g}_i = \partial\boldsymbol{r}^*/\partial\zeta^i, \quad g_{ik} = \boldsymbol{g}_i\cdot\boldsymbol{g}_k, \quad \boldsymbol{g}^i\cdot\boldsymbol{g}_k = \delta^i_k, \\ g^{ik} = \boldsymbol{g}^i\cdot\boldsymbol{g}^k, \quad g = \det g_{ik}, \quad g^{\frac{1}{2}} = [\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3] > 0, \end{gathered} \tag{A 1}$$

where δ^i_k is the Kronecker delta. A body is moving in this space and in the configuration at time t the velocity at the place $\boldsymbol{r}^*$ is

$$\boldsymbol{v}^* = \boldsymbol{v}^*(\zeta^i, t) = v^{*i}\boldsymbol{g}_i. \tag{A 2}$$

For convenience we adopt the notation $\zeta^3 = \zeta$. The equation $\zeta = 0$ defines a fixed surface and points on this surface, together with base vectors, metric tensors and unit normals are specified by

$$\boldsymbol{r} = \boldsymbol{r}(\zeta^\alpha) = \boldsymbol{r}^*(\zeta^1, \zeta^2, 0), \quad \boldsymbol{a}_\alpha = \partial\boldsymbol{r}/\partial\zeta^\alpha, \quad a_{\alpha\beta} = \boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta, \quad a^{\alpha\beta} = \boldsymbol{a}^\alpha\cdot\boldsymbol{a}^\beta,$$
$$\boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta = \delta^\alpha_\beta, \quad a = \det a_{\alpha\beta}, \quad a^{\frac{1}{2}}\boldsymbol{a}_3 = \boldsymbol{a}_1\times\boldsymbol{a}_2, \tag{A 3}$$

where Greek indices have values 1, 2 and all quantities in (A 3) are functions of ζ^1, ζ^2.

The body is assumed to be bounded by the two surfaces

$$\zeta = \zeta_1(\zeta^1, \zeta^2, t), \quad \zeta = \zeta_2(\zeta^1, \zeta^2, t) \tag{A 4}$$

which are smooth and non-intersecting. These surfaces are material surfaces and move with the continuum so that, at these surfaces,

$$\partial\zeta_1/\partial t = v^{*3} - v^{*\alpha}\,\partial\zeta_1/\partial\zeta^\alpha \quad (\zeta = \zeta_1),$$
$$\partial\zeta_2/\partial t = v^{*3} - v^{*\alpha}\,\partial\zeta_2/\partial\zeta^\alpha \quad (\zeta = \zeta_2). \tag{A 5}$$

Let $\mathscr{P}$ be an arbitrary fixed surface bounded by a closed curve $\partial\mathscr{P}$ on the fixed surface $\zeta = 0$ and suppose that $\partial\mathscr{P}$ is defined by the equations

$$f(\zeta^1, \zeta^2) = 0, \quad \zeta = 0. \tag{A 6}$$

The stress vector $\boldsymbol{t}$ across a surface whose unit normal is $\boldsymbol{\nu}^*$ is given by

$$\boldsymbol{t} = \boldsymbol{T}\boldsymbol{\nu}^*, \quad \boldsymbol{T} = \boldsymbol{t}^i\otimes\boldsymbol{g}_i, \quad \boldsymbol{t}^i = \boldsymbol{T}\boldsymbol{g}^i, \tag{A 7}$$

where $\boldsymbol{T}$ is the stress tensor. Let ρ^* be the density of the body and $\boldsymbol{f}^*$ the body force per unit mass. The equations of mass conservation and motion are

$$\partial\rho^*/\partial t + g^{-\frac{1}{2}}(\rho^* g^{\frac{1}{2}} v^{*i})_{,i} = 0, \tag{A 8a}$$
$$\rho^*(\partial\boldsymbol{v}^*/\partial t + v^{*i}\,\partial\boldsymbol{v}^*/\partial\zeta^i) = \rho^*\boldsymbol{f}^* + \operatorname{div}\boldsymbol{T}, \tag{A 8b}$$
$$\boldsymbol{T}^T = \boldsymbol{T}, \quad \operatorname{div}\boldsymbol{T} = g^{-\frac{1}{2}}(g^{\frac{1}{2}}\boldsymbol{t}^i)_{,i}, \tag{A 8c}$$

where $\partial/\partial t$ and $(\ )_{,i} = \partial(\ )/\partial\zeta^i$ are partial derivatives with respect to t and ζ^i.

Next, we multiply (A 8a) by $\lambda_M(\zeta)\,\lambda_N(\zeta)$ $(M = 0, 1, 2, \ldots, K;\ N = 0, 1, 2, \ldots, K)$, where $\lambda_N(\zeta)$ are functions of ζ to be specified and $\lambda_0(\zeta) = 1$, and integrate throughout the volume bounded by the surfaces

$$f(\zeta^1, \zeta^2) = 0, \quad \zeta = \zeta_1, \quad \zeta = \zeta_2. \tag{A 9}$$

The resulting equation, after also using the divergence theorem and the surface conditions, (A 5) yields

$$\frac{\partial}{\partial t}\int_{\mathscr{P}} \rho y_{MN}\,\mathrm{d}\sigma + \int_{\partial\mathscr{P}} \rho\boldsymbol{v}_{MN}\cdot\boldsymbol{\nu}\,\mathrm{d}s - \int_{\mathscr{P}} \rho(\boldsymbol{v}_{MN} + \boldsymbol{v}_{NM})\cdot\mathrm{d}\boldsymbol{\sigma} = 0, \tag{A 10}$$

where $\boldsymbol{\nu} = \nu_\alpha\boldsymbol{a}^\alpha$ is the outward unit normal to $\partial\mathscr{P}$ in the surface $\zeta = 0$, and $\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{a}_3\,\mathrm{d}\sigma$. Also, the velocity fields $\boldsymbol{v}_{MN}$ when referred to the surface base vectors $\boldsymbol{a}_i = (\boldsymbol{a}_\alpha, \boldsymbol{a}_3)$ and the inertia coefficients are

$$\rho a^{\frac{1}{2}} y_{MN} = \int_{\zeta_1}^{\zeta_2} \rho^* g^{\frac{1}{2}} \lambda_M(\zeta)\, \lambda_N(\zeta)\, d\zeta, \quad y_{MN} = y_{NM},$$

$$\rho a^{\frac{1}{2}} \boldsymbol{v}_{MN} = \boldsymbol{a}_\alpha \int_{\zeta_1}^{\zeta_2} \rho^* g^{\frac{1}{2}} \lambda_M(\zeta)\, \lambda_N(\zeta)\, v^{*\alpha}\, d\zeta + \boldsymbol{a}_3 \int_{\zeta_1}^{\zeta_2} \rho^* g^{\frac{1}{2}} \lambda'_M(\zeta)\, \lambda_N(\zeta)\, v^{*3}\, d\zeta, \qquad \text{(A 11)}$$

where a superposed prime on $\lambda_M(\zeta)$ denotes differentiation with respect to ζ.

From (A 8*a*, *b*) we have

$$(\partial/\partial t)\,(\rho^* \lambda_N \boldsymbol{v}^*) + g^{-\frac{1}{2}} \{\rho^* g^{\frac{1}{2}} \lambda_N \boldsymbol{v}^* v^{*i}\}_{,i} - \rho^* \lambda'_N \boldsymbol{v}^* v^{*3} = \lambda_N(\rho^* \boldsymbol{f}^* + \operatorname{div} \boldsymbol{T}). \qquad \text{(A 12)}$$

If we integrate (A 12) throughout the volume bounded by the surfaces (A 9), and use also the surface conditions (A 5) and the divergence theorem, as well as the representation

$$\boldsymbol{v}^* = \sum_{M=0}^{K} \lambda_M(\zeta)\, \boldsymbol{w}_M, \quad \boldsymbol{w}_0 = \boldsymbol{v}, \quad \boldsymbol{w}_M = \boldsymbol{w}_M(\zeta^1, \zeta^2, t), \qquad \text{(A 13)}$$

we obtain

$$\frac{\partial}{\partial t} \int_{\mathscr{P}} \rho \sum_{M=0}^{K} y_{MN}\, \boldsymbol{w}_M\, d\sigma + \int_{\partial\mathscr{P}} \rho \sum_{M=0}^{K} \boldsymbol{w}_M\, \boldsymbol{v}_{MN} \cdot \boldsymbol{\nu}\, ds - \int_{\mathscr{P}} \rho \sum_{M=0}^{K} \boldsymbol{w}_M\, \boldsymbol{v}_{NM} \cdot d\boldsymbol{\sigma}$$

$$= \int_{\mathscr{P}} \rho(\boldsymbol{l}_N - \boldsymbol{k}_N)\, d\sigma + \int_{\partial\mathscr{P}} \boldsymbol{m}_N\, ds \qquad \text{(A 14)}$$

for $N = 0, 1, 2, \ldots, K$. In (A 14),

$$\boldsymbol{m}_N = \boldsymbol{M}_N{}^\alpha \nu_\alpha, \quad \boldsymbol{m}_0 = \boldsymbol{n}, \quad \boldsymbol{M}_0{}^\alpha = \boldsymbol{N}^\alpha, \quad \boldsymbol{l}_0 = \boldsymbol{f} \qquad \text{(A 15)}$$

and

$$\boldsymbol{M}_N{}^\alpha a^{\frac{1}{2}} = \int_{\zeta_1}^{\zeta_2} g^{\frac{1}{2}} \lambda_N \boldsymbol{t}^\alpha\, d\zeta, \quad \boldsymbol{k}_N a^{\frac{1}{2}} = \int_{\zeta_1}^{\zeta_2} g^{\frac{1}{2}} \lambda'_N \boldsymbol{t}^3\, d\zeta, \qquad \text{(A 16)}$$

$$\rho \boldsymbol{l}_N a^{\frac{1}{2}} = \int_{\zeta_1}^{\zeta_2} \rho^* g^{\frac{1}{2}} \lambda_N \boldsymbol{f}^*\, d\zeta + [\boldsymbol{t} g^{\frac{1}{2}} \lambda_N f(\zeta)]_{\zeta=\zeta_1} + [\boldsymbol{t} g^{\frac{1}{2}} \lambda_N f(\zeta)]_{\zeta=\zeta_2}, \qquad \text{(A 17)}$$

where

$$f(\zeta) = \{(\zeta_{,1})^2 g^{11} + (\zeta_{,2})^2 g^{22} + g^{33} + 2(\zeta_{,1}\zeta_{,2} g^{12} - \zeta_{,1} g^{13} - \zeta_{,2} g^{23})\}^{\frac{1}{2}}. \qquad \text{(A 18)}$$

Finally, we note that in the three-dimensional theory, given the representation (A 13), the condition of incompressibility is

$$\left\{\sum_{M=0}^{K} g^{\frac{1}{2}} \lambda_M(\zeta)\, \boldsymbol{w}_M \cdot \boldsymbol{g}^i\right\}_{,i} = 0. \qquad \text{(A 19)}$$

Also, if points $\boldsymbol{r}^*$ in the material body at time t are represented by

$$\boldsymbol{r}^* = \boldsymbol{r} + \sum_{N=1}^{K} \lambda_N(\zeta)\, \bar{\boldsymbol{d}}_N,$$

then since

$$\int_{\zeta_1}^{\zeta_2} \rho^* v^{*i} \boldsymbol{g}_i \times \boldsymbol{v}^*\, d\zeta = 0,$$

it follows that

$$\sum_{M=0}^{K} \sum_{N=0}^{K} (\boldsymbol{a}^\alpha \cdot \boldsymbol{v}_{MN}\, \bar{\boldsymbol{d}}_{N,\alpha} + \boldsymbol{a}_3 \cdot \boldsymbol{v}_{NM}\, \boldsymbol{d}_N) \times \boldsymbol{w}_M = \boldsymbol{0}. \qquad \text{(A 20)}$$

References

Benjamin, T. B. & Feir, J. E. 1967 The disintegration of wave trains on deep water. I. Theory. *J. Fluid Mech.* **27**, 417–430.

Boussinesq, J. 1871 Théorie de l'intumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire. *C.r. hebd. Séanc. Acad. Sci., Paris* **72**, 755–759.

Caulk, D. A. 1976 On the problem of fluid flow under a sluice gate. *Int. J. Engng Sci.* **14**, 1115–1125.

Cumberbatch, E. 1958 Two-dimensional planing at high Froude number. *J. Fluid Mech.* **4**, 466–478.

Ertekin, R. C., Webster, W. C. & Wehausen, J. V. 1984 Ship-generated solitons. *Proceedings 15th symposium on naval hydrodynamics (Hamburg)*, pp. 1–15.

Green, A. E. 1935 The gliding of a plate on a stream of finite depth. *Proc. Camb. phil. Soc.* **31**, 589–603.

Green, A. E. 1936*a* The gliding of a plate on a stream of finite depth. II. *Proc. Camb. phil. Soc.* **32**, 67–85.

Green, A. E. 1936*b* Note on the gliding of a plate on the surface of a stream. *Proc. Camb. phil. Soc.* **32**, 248–252.

Green, A. E. 1938 The moment of the fluid forces acting on a plate which is gliding on the surface of a stream. *Proc. Camb. phil. Soc.* **34**, 167–184.

Green, A. E., Laws, N. & Naghdi, P. M. 1974 On the theory of water waves. *Proc. R. Soc. Lond.* A **338**, 43–55.

Green, A. E. & Naghdi, P. M. 1976*a* Directed fluid sheets. *Proc. R. Soc. Lond.* A **347**, 447–473.

Green, A. E. & Naghdi, P. M. 1976*b* A derivation of equations for wave propagation in water of variable depth. *J. Fluid Mech.* **78**, 237–246.

Green, A. E. & Naghdi, P. M. 1977 Water waves in a nonhomogeneous incompressible fluid. *J. appl. Mech.* **44**, 523–528.

Green, A. E. & Naghdi, P. M. 1983 On electromagnetic effects in the theory of shells and plates. *Phil. Trans. R. Soc. Lond.* A **309**, 559–610.

Green, A. E. & Naghdi, P. M. 1984 A direct theory of viscous fluid flow in channels. *Arch. ration. Mech. Analysis* **86**, 39–63.

Naghdi, P. M. & Rubin, M. B. 1981*a* On the transition to planing of a boat. *J. Fluid Mech.* **103**, 345–374.

Naghdi, P. M. & Rubin, M. B. 1981*b* On inviscid flow in a waterfall. *J. Fluid Mech.* **103**, 375–387.

Naghdi, P. M. & Rubin, M. B. 1982 On the effect of curvature at the detachment point of a fluid sheet from a rigid boundary. *Physics Fluids* **25**, 1110–1116.

Naghdi, P. M. & Rubin, M. B. 1984 On the squat of a ship. *J. Ship Res.* **28**, 107–117.

Rayleigh, Lord 1876 On waves. *Phil. Mag.* (5) **1**, 257–279 (Papers *1*, 251–271).

Squire, H. B. 1957 The motion of a simple wedge along the water surface. *Proc. R. Soc. Lond.* A **243**, 48–64.

Vanden-Broeck, J. M. 1980 Nonlinear stern waves. *J. Fluid Mech.* **96**, 603–611.

Wagner, H. 1932 Über Stross- und Gleit vorgänge an der Oberfläche von Flüssigkeiten. *Z. angew. Math. Mech.* **12**, 193–215.

Whitham, G. M. 1974 *Linear and nonlinear waves*. New York, London, Sydney, Toronto: Wiley and Sons.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF